Remark: Radius of Conv.

of $\sum_{n=0}^{\infty} C_n (x-a)^n = R \in [0,\infty]$

$\Leftrightarrow \begin{cases} \text{convergent (absolutely) on } |x-a| < R \\ \text{diverges } |x-a| > R \end{cases}$

Remark If $\lim_{n\to\infty} |C_n|^{\frac{1}{n}}$ exists $(\in[0,\infty])$

(or $\lim \left|\frac{C_{n+1}}{C_n}\right|$)

Then $R = 1 / \lim_{n\to\infty} |C_n|^{\frac{1}{n}}$ (or $1/\lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|}$) $\left(\frac{1}{0}=\infty, \ \frac{1}{\infty}=0\right)$

Since $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n \begin{cases} \text{conv. absolutely} \\ \text{div} \end{cases}$

if $\displaystyle\lim_{n\to\infty} |u_n|^{\frac{1}{n}} = \lim_{n\to\infty} |c_n|^{\frac{1}{n}} |x-a| \begin{matrix} < 1 \\ > 1 \end{matrix}$

(Similarly for ratio test)

<u>Remark</u> R always exists

but $\displaystyle\lim_{n\to\infty} |c_n|^{\frac{1}{n}}$ or $\displaystyle\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}$

need not exist.

Example: $\displaystyle\sum_{n=1}^{\infty} a_n x^n = \left(\frac{x}{2}\right)^1 + \left(\frac{x}{4}\right)^2 + \left(\frac{x}{2}\right)^3 + \left(\frac{x}{4}\right)^4 + \ldots$

(Beyond this course)

$\limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$ always exists and $R = \frac{1}{(\ \ )} \in [0, \infty]$

In previous example

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{2}$$

$\therefore R = 2$

Algebraic operations

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$B(x) = \sum_{n=0}^{\infty} b_n x^n$$

both conv. (abs) on $|x| < R$

$$A(x) \cdot B(x) = ?$$

$$\begin{array}{rl} & A(x) = a_0 + a_1 x + a_2 x^2 + \cdots \\ \times) & B(x) = b_0 + b_1 x + b_2 x^2 + \cdots \\ \hline \end{array}$$

$$A(x)B(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \cdots$$

Thm If $A(x)=\sum_{n=0}^{\infty} a_n x^n$, $B(x)=\sum_{n=0}^{\infty} b_n x^n$

both conv. (abs.) on $|X|<R$

and $C_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$,

Than $\sum_{n=0}^{\infty} C_n x^n$ also conv (abs)

on $|x|<R$ and $= A(x)B(x)$

Reason. If $\lim |a_n|^{\frac{1}{n}} = \underline{\lim |b_n|^{\frac{1}{n}}} = \frac{1}{R}$

(not rigorous) (may not exist)

than $|a_n| \sim \left(\frac{1}{R}\right)^n$, $|b_n| \sim \left|\frac{1}{R}\right|^n$,

$\Rightarrow |C_n| \lesssim (n+1)\left(\frac{1}{R}\right)^n \Rightarrow \lim |C_n|^{\frac{1}{n}} \sim \frac{1}{R}$

Example. (We will see that for $x \in \mathbb{R}$, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ later)

$$A(x) = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$B(x) = 1 - \frac{x}{1} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

Firstly, $R(A(x)) = \infty = R(B(x))$ (ratio or root test)

$$\sum_{n=0}^{\infty} c_n x^n. \quad c_0 = 1$$

$$c_1 = -1 + 1 = 0$$

$$c_2 = \frac{1}{2} - 1 + \frac{1}{2} = 0$$

$$c_3 = \frac{-1}{6} + \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = 0$$

Here we compute c0, c1, c2, ... for power series of A(x)B(x)

How about Power series

for $\frac{A(x)}{B(x)} = ?$

Assume $\frac{A(x)}{B(x)} = \sum_{n=0}^{\infty} c_n x^n$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = \left(\sum_{n=0}^{\infty} b_n x^n\right)\left(\sum_{n=0}^{\infty} c_n x^n\right)$$

$$1 = a_0 = b_0 \underline{c_0} = 1 \cdot \underline{c_0}, \quad c_0 = 1$$

$$1 = a_1 = b_0 \underline{c_1} + b_1 c_0, \quad c_1 = 2$$

$$\frac{1}{2} = a_2 = b_0 \underline{c_2} + b_1 c_1 + b_2 c_0, \quad c_2 = 2$$

Here we compute c0, c1, c2 for power series of A(x)/B(x)

Term by term differentiation

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) = ?$$

<u>Theorem</u> If $f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$

conv. (abs) on $|x-a| < R$

Then (1) $f', f'', \cdots f^{(n)}$ all exist on $|x-a| < R$

(2) $f'(x) = \sum_{n=1}^{\infty} n C_n (x-a)^{n-1}$,

$f''(x) = \sum_{n=2}^{\infty} n(n-1) C_n (x-a)^{n-2}$,

$\cdots$ on $|x-a| < R$.

Ex $f(x) = 1 + x + x^2 + \cdots$

$= \frac{1}{1-x}$ on $|x| < 1$

$\overset{\frac{d}{dx}}{\Rightarrow}$ $1 + 2x + 3x^2 + \cdots = \frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}$

on $|x| < 1$

Remark $\sum_{n=1}^{\infty} \frac{\sin(n!x)}{n^2}$ conv.

abs for any $x \in \mathbb{R}$, but

$\sum_{n=1}^{\infty} \frac{n!\cos(n!x)}{n^2}$ div.

for any $x \in \mathbb{R}$

<u>Thm</u> (term by term integration)

If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$

conv. abs. on $|x-a| < R$.

Then $\sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1}$

also conv. abs. on $|x-a| < R$

and $\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1} + C$

$$\left(\boxed{\int_a^x f(t)\,dt = \int_a^x c_n (t-a)^n\,dt = \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1}} \right.$$

Example $F(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

on $|x|<1$, What is $f(x)$?

Ans (1) Σ conv. abs on $|x|<1$

(2) $F'(x) = 1 - x^2 + x^4 - \cdots$

$= \frac{1}{1+x^2}$ on $|x|<1$

$\therefore F(x) = \int_0^x F'(t)\,dt = \int_0^x \frac{1}{1+t^2}\,dt$

$= \tan^{-1} t \Big|_0^x = \tan^{-1} x$ on $|x|<1$