Def: Power Series:

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

a: Center

c_n: Coefficients

The central issue here is to find out for what $x \in \mathbb{R}$ is the series convergent.

Example $\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n (x-2)^n$

Geometric Series.

$r = \frac{-(x-2)}{2}$

converges when $|r| < 1$

i.e. $0 < x < 4$

Rm (1) it converges absolutely on $0 < x < 4$.

(2) it diverges on $\left|\frac{x-2}{2}\right| \geq 1$

Example $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Ratio test on

$$|u_n| = \frac{|x|^n}{n!}$$

$$\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = 0$$

$\therefore$ converges absolutely for any $x \in \mathbb{R}$

E

Example $\sum_{n=0}^{\infty} n!\, x^n$

Ratio test on $|u_n|$

$$\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \begin{cases} \infty, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\therefore \lim_{n\to\infty} |u_n| \neq 0, \text{ if } x \neq 0$$

$\sum_{n=0}^{\infty} n!\, x^n$ diverges

for any $x \neq 0$

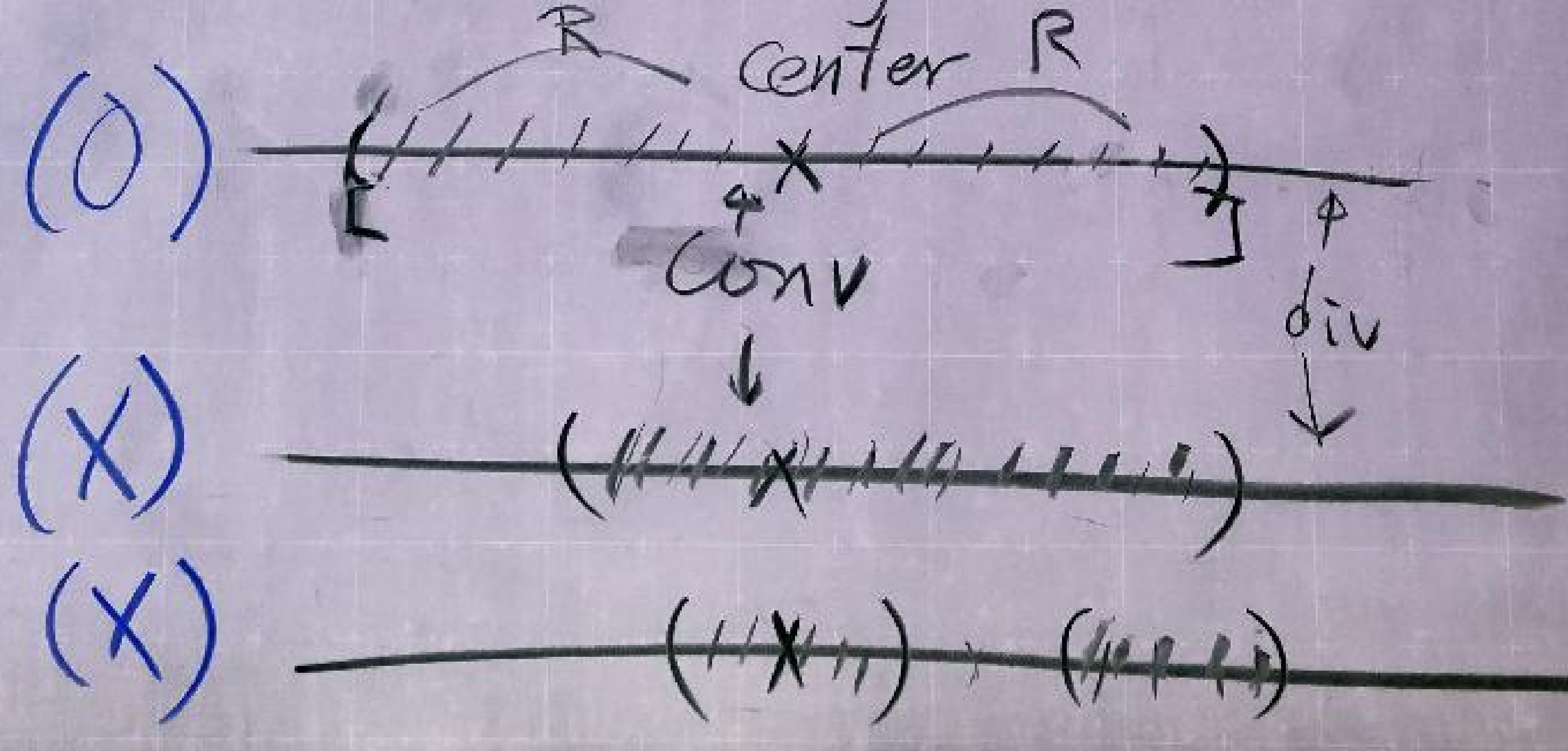
Theorem If $\sum_{n=0}^{\infty} a_n x^n$ conv.

at $x = c \neq 0$, then it

converges absolutely for $|x| < |c|$

($\Longrightarrow$ If it diverges at $x = d$

then it diverges for $|x| > |d|$)



Pf If $\sum_{n=0}^{\infty} a_n c^n$ converges

$\Rightarrow \lim_{n\to\infty} |a_n c^n| = 0$

$\Rightarrow |a_n c^n| < 1, \ \forall n \geq N$

$\Rightarrow |a_n| < \frac{1}{|c|^n}, \ \forall n \geq N$

If $|x| < |c|$

$$\sum_{n=N}^{\infty} |a_n x^n| < \sum_{n=N}^{\infty} \left|\frac{x}{c}\right|^n < \infty$$

i.e. converges absolutely on $|x| < |c|$

Remark Possible cases

(1) Converges for all $x \in \mathbb{R}$

(Radius of conv. $R = \infty$)

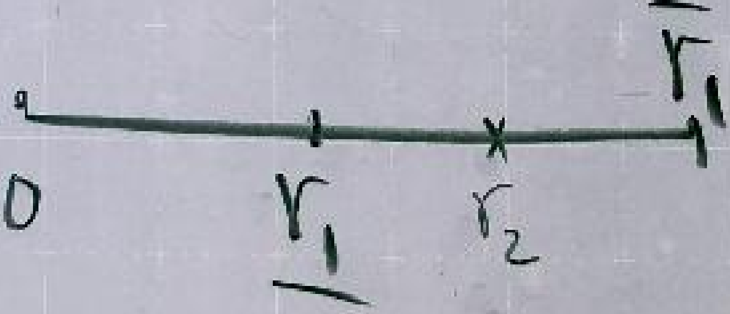
(2) Converges only at $x = 0$

(Radius of conv. $R = 0$)

(3) If converge at $x = \underline{r}_1$

diverges at $x = \overline{r}_1$

$0 < \underline{r}_1 < \overline{r}_1$



Let $r_2 = \frac{\underline{r}_1 + \overline{r}_1}{2}$

If conv at r_2, let $\underline{r}_2 = r_2$, $\overline{r}_2 = \overline{r}_1$

If div at r_2, let $\underline{r}_2 = \underline{r}_1$, $\overline{r}_2 = r_2$

Continue on

Either $\underline{r}_{n+1} = \dfrac{\underline{r}_n + \overline{r}_n}{2}$, $\overline{r}_{n+1} = \overline{r}_n$

or $\underline{r}_{n+1} = \underline{r}_n$, $\overline{r}_{n+1} = \dfrac{\underline{r}_n + \overline{r}_n}{2}$

$\underline{r} \quad \cdots \le \overline{r}_n \cdots \le \overline{r}_2 \le \overline{r}_1$

R

$\underline{r}_1 \le \underline{r}_2 \le \underline{r}_n \le \cdots \qquad \overline{r}_1$

From monotonic seq. Thm

$$\lim_{n\to\infty} \underline{r}_n = \lim_{n\to\infty} \overline{r}_n = R \quad \left(\begin{array}{c} 0 < R < \infty \\ \text{radius of} \\ \text{convergence} \end{array} \right)$$

$$\boxed{\Rightarrow \sum_{n=0}^{\infty} a_n x^n \ \begin{array}{ll} \text{conv. abs.} & \text{for } |x| < R \\ \text{div.} & \text{for } |x| > R \end{array}}$$

Rm If $0 < R < \infty$

Then $\sum_{n=0}^{\infty} a_n x^n$ can either conv.

or diverge at $x = \pm R$.

Example ⓐ $\sum_{n=0}^{\infty} x^n$ conv. on $(-1, 1)$

div. elsewhere.

ⓑ $\sum_{n=0}^{\infty} \frac{x^n}{n}$ conv, on $[-1, 1)$

div elsewhere

(ratio or root test on $|u_n|$)

ⓒ $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$ conv. on $[-1, 1]$

div. elsewhere.