Alternating Series

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n, \quad u_n > 0$$

Example: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Leibnitz Test

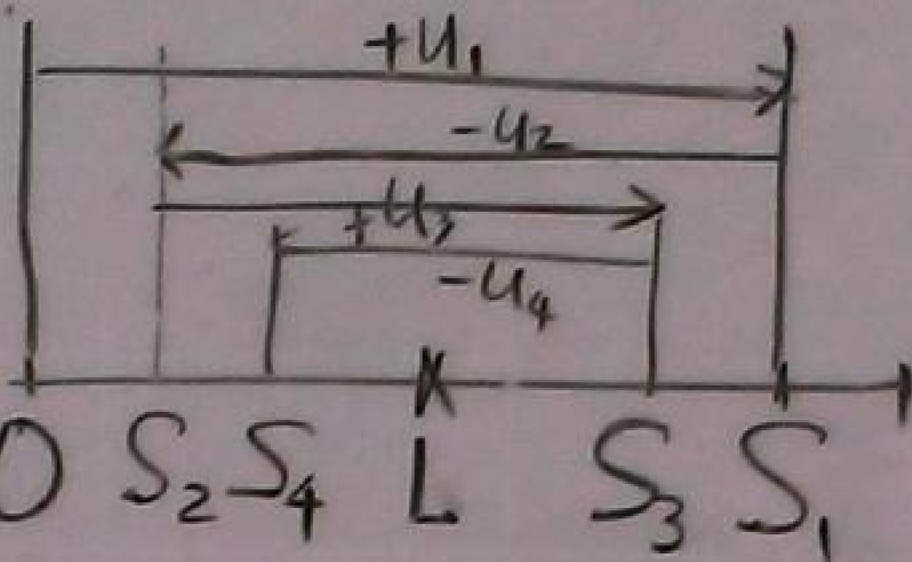
If (1) $u_n > 0$

(2) $u_n \geq u_{n+1}$ for all $n \geq N$

(3) $u_n \longrightarrow 0$

Then $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ converges

Pf: $S_n = \sum_{k=1}^{n} (-1)^{k+1} u_k$



$$0 < S_2 < S_4 < \cdots < \underset{\substack{\| \\ u_1}}{S_1}$$

$$S_1 > S_3 > \cdots > 0$$

$\therefore \{S_2, S_4, S_6, \cdots\}$

is an increasing seq.

and bounded above

$$\therefore \lim_{n\to\infty} S_{2n} = L$$

$$S_{2n+1} = S_{2n} + u_{2n+1}$$

$$\to L + 0 = L$$

$$\therefore \lim_{n\to\infty} S_n = L$$

<u>Remark:</u>

(1) $S_{2m} < L < S_{2n+1}$

for any $m, n \in \mathbb{N}$

(2) $|L - S_n| < u_{2n+1}$

$$\left(\begin{array}{l} 0 < L - S_{2m} < u_{2m+1} \\ -u_{2n} < L - S_{2n-1} < 0 \end{array} \right)$$

Def $\sum_{n=1}^{\infty} a_n$ converges absolutely

if $\sum_{n=1}^{\infty} |a_n| < \infty$

Def $\sum_{n=1}^{\infty} a_n$ conv. conditionally

if $\begin{cases} \sum_{n=1}^{\infty} a_n \text{ converges} \\ \sum_{n=1}^{\infty} |a_n| = \infty \end{cases}$

Example (a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

converges conditionally

(b) $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ conv. absolutely

Theorem If $\sum_{n=1}^{\infty}|a_n| < \infty$

then $\sum_{n=1}^{\infty} a_n$ converges

Pf: $-|a_n| \leq a_n \leq |a_n|$

$$0 \leq \underbrace{a_n + |a_n|}_{\text{non-negative}} \leq \underbrace{2|a_n|}_{\text{conv.}}$$

$\therefore \sum_{n=1}^{\infty}(a_n + |a_n|)$ converges

$$\sum_{n=1}^{\infty} a_n = \underbrace{\sum_{n=1}^{\infty}(a_n + |a_n|)}_{\text{conv.}} - \underbrace{\sum_{n=1}^{\infty}|a_n|}_{\text{conv}}$$

$=$ convergent.

Example ⓑ $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges.

$\because$ it converges absolutely.

Example. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} \quad p>0$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{conv.} & p>1 \\ \text{div.} & 0<p\le 1 \end{cases}$$

$$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} \text{ conv.} \begin{cases} \text{absolutely}, & p>1 \\ \text{conditionally} & 0<p\le 1 \end{cases}$$

(Leibnitz test)

Example $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{\ln n}}$

(i) Convergent (Leibnitz test)

(ii) conv. conditionally

$$\sum_{n=2}^{\infty} |a_n| = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{\frac{1}{2}}}$$

Integral test:

$$\int_2^{\infty} \frac{1}{x(\ln x)^{\frac{1}{2}}} dx = \int_{x=2}^{\infty} \frac{d\ln x}{(\ln x)^{\frac{1}{2}}}$$

$$= \int_{y=\ln 2}^{\infty} \frac{dy}{y^{\frac{1}{2}}} = \infty$$

<u>Remark</u>: $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}\cdots$

converges and >0

but its rearrangent may converge to a different limit:

$$\left(-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}\right) < -1$$

>-1

$>-1 \quad +1$

$>-1 \quad \left(-\frac{1}{10}-\frac{1}{12}-\frac{1}{14}\cdots -1/62\right) \quad <-1 \quad <-1$

$\left(+\frac{1}{3}\right)$

$\left(-1/64 - \dots - 1/120\right)$

$\left(+1/5\right) \cdots \quad \therefore \Sigma(\ ) = -1$

Reason:

If $\sum_{n=1}^{\infty} a_n$ converges conditionally,

then $\begin{cases} \sum_{a_n>0} a_n = \infty \\ \sum_{a_n<0} a_n = -\infty \end{cases}$

Rearrangent of $\sum_{n=1}^{\infty} a_n$

can converge to any L