Limit Comparison test

$a_n > 0,\ b_n > 0 \quad \forall n \geq N$

① If $\lim_{n \to \infty} \frac{a_n}{b_n} = c,\ 0 < c < \infty$

Then $\sum b_n < \infty \Leftrightarrow \sum a_n < \infty$
$\qquad (= \infty) \qquad\qquad (= \infty)$

② If $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$

Then $\sum b_n < \infty \Rightarrow \sum a_n < \infty$

③ If $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$

Then $\sum b_n = \infty \Rightarrow \sum a_n = \infty$

Key: Compare with known Series such as $\sum ar^n$, $\sum \frac{1}{n^p}$

<u>Example</u> $\sum_{n=1}^{\infty} \frac{\ln n}{n^{\frac{3}{2}}} < \infty$

$$\boxed{\underline{\text{Fact}}:\ n^a >> \ln n,\ a>0}$$

$$\because \frac{\ln n}{n^{\frac{3}{2}}} = \left(\frac{1}{n^{\frac{5}{4}}}\right) \cdot \left(\frac{\ln n}{n^{\frac{1}{4}}}\right)$$

<u>Compare with $\sum 1/n^{\frac{5}{4}}$:</u>

$$\lim_{n\to\infty} \frac{\ln n / n^{\frac{3}{2}}}{1 / n^{\frac{5}{4}}} = \lim_{n\to\infty} \frac{\ln n}{n^{\frac{1}{4}}} = 0$$

Examples

(a) $\displaystyle\sum \frac{1}{n+\sqrt{n}}$

(b) $\displaystyle\sum \frac{1}{\sqrt{n^3+2}}$

(c) $\displaystyle\sum \frac{(\ln n)^2}{n^{\frac{3}{2}}}$

Compare with

$\displaystyle\sum \frac{1}{n} = \infty$

$\displaystyle\sum \frac{1}{n^{\frac{3}{2}}} < \infty$

$\displaystyle\sum \frac{1}{n^{\frac{5}{4}}} < \infty$

$$\therefore \frac{(\ln n)^2}{n^{\frac{3}{2}}} = \frac{1}{n^{\frac{5}{4}}}\left(\frac{\ln n}{n^{\frac{1}{8}}}\right)^2$$

d $\sum \tan \frac{1}{n^2}$ $\Big|$ Compare with $\sum \frac{1}{n^2} < \infty$

$$\therefore \lim_{x \to 0} \frac{x}{\tan x} = \lim_{x \to 0} \frac{1}{\sec^2 x} = 1$$

e $\sum \frac{1}{n\sqrt[n]{n}}$ $\Big|$ Compare with $\sum \frac{1}{n} = \infty$

$$\lim_{n \to \infty} \frac{1/n}{1/(n\sqrt[n]{n})} = \lim_{n \to \infty} \sqrt[n]{n} = 1$$

(f) $\sum \sqrt{\frac{\ln n}{n}} > \sum \frac{1}{n^{\frac{1}{2}}} = \infty$

(g) $\sum \frac{1}{\sqrt{n \ln n}} = \infty$

$\because = \sum \frac{1}{n^{\frac{3}{4}}} \left(\frac{n^{\frac{1}{4}}}{\sqrt{\ln n}} \right)$

$\lim_{n \to \infty} \frac{n^{\frac{1}{4}}}{\sqrt{\ln n}} = \infty$

Compare with $\sum \frac{1}{n^{\frac{3}{4}}} = \infty$

Ratio test:

If $a_n > 0$, $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \rho$

ⓐ $\sum a_n < \infty$ if $\rho < 1$

ⓑ $\sum a_n = \infty$ if $\rho > 1$

ⓒ Inconclusive if $\rho = 1$

$$\left(\begin{array}{l} \sum \frac{1}{n} = \infty, \quad \rho = 1 \\ \sum \frac{1}{n^2} < \infty, \quad \rho = 1 \end{array} \right)$$

proof: ⓐ $\lim \frac{a_{n+1}}{a_n} = \rho < 1$

take $r = \frac{\rho+1}{2}$, $\rho < r < 1$

$r = \rho + \underbrace{\left(\frac{1-\rho}{2}\right)}_{=\varepsilon}$

Since $\lim \cancel{a_{n+1}/a_n} = \rho$

$\Rightarrow \frac{a_{n+1}}{a_n} - \rho < \varepsilon$

for all $n \geq N$

$\therefore a_{N+1} < (\rho+\varepsilon) a_N = r a_N$

$\therefore a_{N+2} < r a_{N+1} < r^2 a_N$

$\therefore a_{N+k} < r^k a_N$

$$\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

$$< \left(\sum_{n=1}^{N} a_n\right) + r a_N + r^2 a_N + \cdots$$

$$= \left(\sum_{n=1}^{N} a_n\right) + \text{convergent geometric series}$$

$$< \infty$$

(b) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho > 1$

$$\frac{a_{n+1}}{a_n} > \frac{\rho+1}{2}, \quad \left(1 < \frac{\rho+1}{2} < \rho\right)$$

for all $n \geq N$. $\therefore \lim_{n\to\infty} a_n \neq 0$

Examples

(a) $\sum_{n=1}^{\infty} \frac{2^n+5}{3^n}$,

$$a_n = \frac{2^n+5}{3^n}$$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{2}{3}$$

$\therefore$ Convergent

(b) $\sum \frac{(2n)!}{n!\,n!}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(2n+2)!}{(n+1)!\,(n+1)!}}{\frac{(2n)!}{n!\,n!}}$$

$$= \frac{(2n+2)(2n+1)}{(n+1)(n+1)}$$

$$\therefore \rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 4$$

divergent

ⓒ $\sum \frac{4^n n! n!}{(2n)!}$

from ⓑ

we know $\rho = 1$ here.

No conclusion yet

$$\frac{a_{n+1}}{a_n} = \frac{4(n+1)(n+1)}{(2n+2)(2n+1)}$$

$$= \left(\frac{n+1}{n+\frac{1}{2}}\right) > 1 \quad \therefore a_{n+1} > a_n$$

$\therefore \lim_{n\to\infty} a_n \neq 0$, divergent

(c) $\sum \frac{(2n)!}{4^n n! n!} = ?$

See homework 2 for details

$\lim \frac{a_{n+1}}{a_n} = \frac{n+\frac{1}{2}}{n+1} = \frac{2n+1}{2n+2} < 1$

Root test:

Example: $a_n = \begin{cases} n/2^n, & n \text{ odd} \\ 1/2^n, & n \text{ even} \end{cases}$

$\frac{a_{n+1}}{a_n} = \begin{cases} \frac{1}{2n} & n \text{ odd} \\ \frac{n+1}{2} & n \text{ even} \end{cases}$

Ratio test is not applicable.

Root test:

If $a_n \geq 0$ for all $n \geq N$

and $\lim_{n\to\infty} a_n^{\frac{1}{n}} = \rho$

(a) $\sum a_n < \infty$ if $\rho < 1$

(b) $\sum a_n = \infty$ if $\rho > 1$

(c) Inconclusive if $\rho = 1$

$$\left(\begin{array}{l} \sum \frac{1}{n} = \infty, \ \rho = 1 \\ \sum \frac{1}{n^2} < \infty, \ \rho = 1 \end{array} \right)$$

pf (a) $r = \frac{\rho+1}{2}$, $\rho < r < 1$

$$\lim_{n\to\infty} a_n^{\frac{1}{n}} = \rho$$

$\Rightarrow$ $a_n^{\frac{1}{n}} < r$ for all $n > N$

$\therefore$ $a_n < r^n$ for all $n > N$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

$$< \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} r^n < \infty$$

(b) Similar to ratio test

Root test

Examples

(a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$, $\rho = \frac{1}{2}$

conv.

(b) $a_n = \begin{cases} \frac{n}{2^n}, & n \text{ odd} \\ \frac{1}{2^n}, & n \text{ even} \end{cases}$

$\rho = \frac{1}{2}$, conv. (root test)

(c) $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ $\rho = 2$, divergent (root test)

(d) $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$, $\rho = 0$, convergent.