Infinite Series $\sum_{n=1}^{\infty} a_n$

<u>Def</u> $\sum_{n=1}^{\infty} a_n = L$

iff $\lim_{n\to\infty} S_n = L$

where $S_n = \sum_{k=1}^{n} a_k$

<u>Remark</u> If $\sum_{n=1}^{\infty} a_n = {+\infty \atop -\infty}$,

we also call it divergent.

Examples of convergent Series

Geometric Series

$$a+ar+ar^2+\cdots+ar^{n-1}+\cdots$$

$$S_n=a+ar+\cdots+ar^{n-1}$$

$$rS_n=\quad ar+\cdots+ar^{n-1}+ar^n$$

$$(1-r)S_n=a(1-r^n)$$

$$r\neq 1 \Rightarrow S_n=\frac{a(1-r^n)}{1-r}$$

$\therefore$ convergent if $|r|<1$

$\therefore$ divergent if $|r|>1$, $r=-1$

$r=1$, $\therefore$ $a+a+a+\cdots\cdots$ divergent

Examples of divergent Series

Theorem.

$$\sum_{n=1}^{\infty} a_n \text{ conv.} \underset{\not\Leftarrow}{\Rightarrow} \lim_{n\to\infty} a_n = 0$$

$$\left(\sum_{n=1}^{\infty} a_n \text{ div.} \Leftarrow \lim_{n\to\infty} a_n \neq 0 \right)$$

<u>pf</u> $a_n = S_n - S_{n-1}$

If $\lim_{n\to\infty} S_n = S$, $\lim_{n\to\infty} S_{n-1} = S$

$\therefore \lim_{n\to\infty} a_n = S - S = 0$

__Examples__

$$\sum_{n=1}^{\infty} n^2, \quad \sum_{n=1}^{\infty} (-1)^n, \quad \sum_{n=1}^{\infty} \frac{-n}{2n+5}$$

all diverge.

__Example__ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots$$

$$> 1+\frac{1}{2}+\underbrace{\left(\frac{1}{4}+\frac{1}{4}\right)}_{\frac{1}{2}}+\underbrace{\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)}_{\frac{1}{2}}+\cdots$$

$$= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$$

divergent. since $S_n \not\leq M$

for any fixed M

If $a_n \geq 0$, then S_n is nondecreasing ($S_{n-1} \leq S_n$)

<u>Thm</u> If $a_n \geq 0$

then $\sum_{n=1}^{\infty} a_n$ converges

$\Longleftrightarrow$ $S_n \leq M$ for all n

<u>pf</u>: use the theorem (Theorem 6, section 10.1) from last lecture.

How about $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p>0$?

Integral test:

If $a_n > 0$, $a_n = f(n)$

and $f(x)$ is continuous, positive and decreasing for all $x \geq N$.

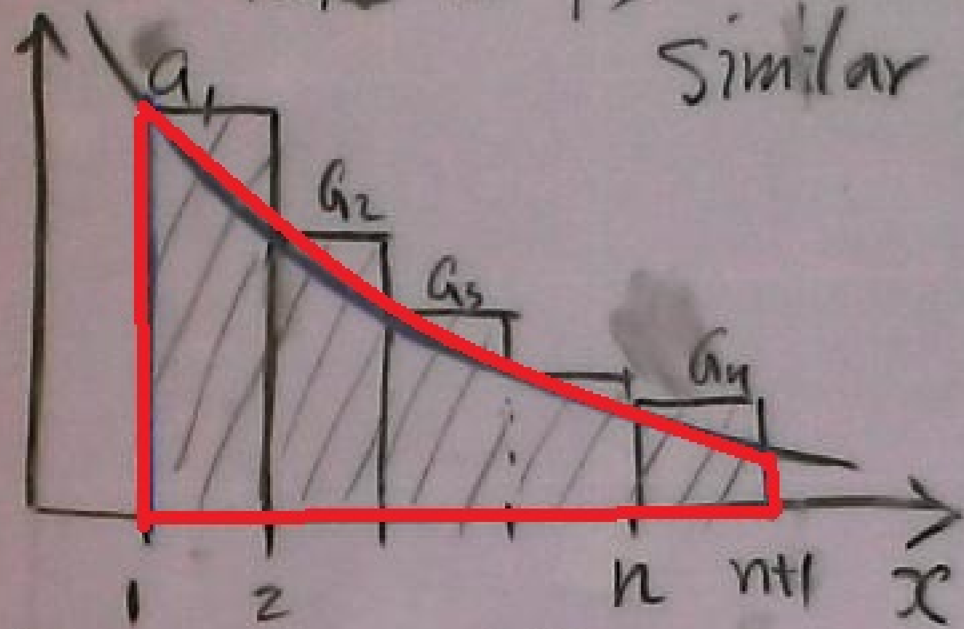
Then

$$\sum_{n=N}^{\infty} a_n \text{ conv.} \iff \int_N^{\infty} f(x)\,dx \text{ conv.}$$

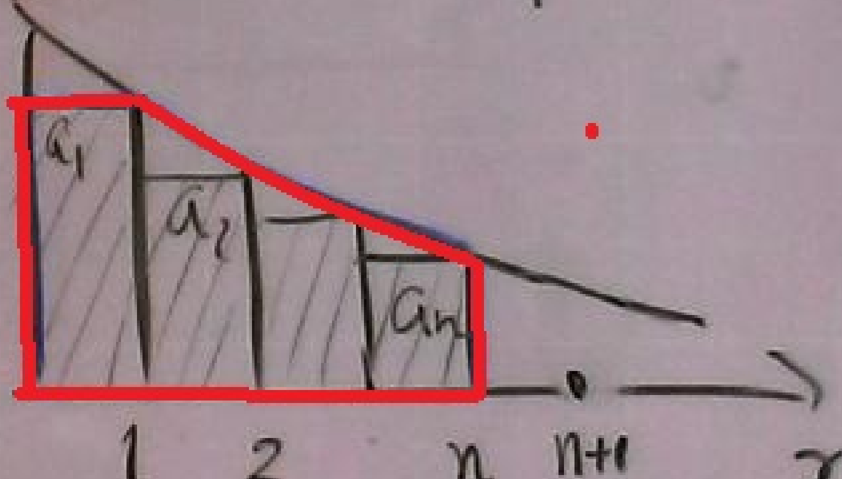
$$\left(\text{Example } \sum_{n=1}^{\infty} \frac{1}{n^p} \iff \int_1^{\infty} \frac{1}{x^p}\,dx \right)$$

pf

Take N=1, otherwise similar



$$\sum_{k=1}^{n} a_k \geq \int_1^{n+1} f(x)\,dx$$



$$\sum_{k=1}^{n} a_k \leq a_1 + \int_1^{n} f(x)\,dx$$

$$\int_1^{n+1} f(x)\,dx \leq \sum_{k=1}^{n} a_k \leq a_1 + \int_1^{n} f(x)\,dx$$

Example

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \text{convergent} & p>1 \\ \text{divergent} & 0<p\le 1 \end{cases}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{conv.} & p>1 \\ \text{div.} & 0<p\le 1 \end{cases}$$

Remark: Similarly, we have

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_n^\infty f(x)\,dx$$

where $R_n = S - S_n = \sum_{k=n+1}^{\infty} a_k$

Remark :

$$\lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx$$

$$= \lim_{b\to\infty}\int_1^b x^{-p}\,dx = \begin{cases} \left.\frac{1}{-p+1}x^{-p+1}\right|_1^b, & p\neq 1 \\ \left.\ln x\right|_1^b, & p=1 \end{cases}$$

$$= \lim_{b\to\infty}\begin{cases} \frac{1}{-p+1}\left(b^{-p+1}-1\right), & p\neq 1 \\ \ln b, & p=1 \end{cases}$$

$$= \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ +\infty & \text{if } p\leq 1 \end{cases}$$

Comparison test:

If $a_n, c_n, d_n \geq 0$

and $d_n \leq a_n \leq c_n \quad \forall n \geq N$

Then

① $\sum_{n=N}^{\infty} c_n$ conv. $(<\infty) \Rightarrow \sum_{n=N}^{\infty} a_n$ conv. $(<\infty)$

② $\sum_{n=N}^{\infty} d_n$ div. $(=\infty) \Rightarrow \sum_{n=N}^{\infty} a_n$ div. $(=\infty)$

Examples

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{5n-1}$

$$\frac{5}{5n} < \frac{5}{5n-1}$$

Since $\sum \frac{1}{n}$ diverges

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{5}{5n-1}$ diverges.

(b) $\sum_{n=0}^{\infty} \frac{1}{n!}$

$= 1 + \frac{1}{1} + \frac{1}{2\cdot 1} + \frac{1}{3\cdot 2\cdot 1} + \frac{1}{4\cdot 3\cdot 2\cdot 1} + \cdots$

$< 1 + \frac{1}{1} + \frac{1}{2\cdot 1} + \frac{1}{2\cdot 2} + \frac{1}{2\cdot 2\cdot 2} + \cdots$

$=$ Convergent

$\therefore \sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

(c) $5+\frac{2}{3}+1+\frac{1}{2+\sqrt{1}}$

$+\frac{1}{4+\sqrt{2}}+\cdots+\frac{1}{2^n+\sqrt{n}}+\cdots$

$\therefore$ Ignore first few terms

and compare with

$\sum\frac{1}{2^n+\sqrt{n}}<\sum\frac{1}{2^n}$ conv.

$\therefore$ Original Series converges.

Limit comparison Theorem

$a_n > 0,\ b_n > 0,\ \forall n \geq N$

① If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c,\ 0 < c < \infty$

Then $\sum\limits_{n=N}^{\infty} a_n < \infty \Leftrightarrow \sum\limits_{n=N}^{\infty} b_n < \infty$

② If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$

Then $\sum\limits_{n=N}^{\infty} b_n < \infty \Rightarrow \sum\limits_{n=N}^{\infty} a_n < \infty$

③ If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$

Then $\sum\limits_{n=N}^{\infty} b_n = \infty \Rightarrow \sum\limits_{n=N}^{\infty} a_n = \infty$

Pf: (Only for case ①)

If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c > 0$

Take ε =c/2 in Definition, p552, then

$\therefore \exists$ integer M such that

for all $n \geq M$

$$\frac{-c}{2} < \frac{a_n}{b_n} - c < \frac{c}{2}$$

$$\therefore \frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2}$$

$$\frac{c}{2} b_n < a_n < \frac{3c}{2} b_n$$

Examples

① $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$

Ans Compare with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

$$\therefore \lim_{n\to\infty} \frac{\frac{2n+1}{(n+1)^2}}{\frac{1}{n}} = 2$$

div.

② $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

Ans: Compare with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$

limit comparison Theorem $\Rightarrow$ conv.

③ $$\sum_{n=1}^{\infty} \frac{1+n\ln n}{n^2+5}$$

<u>Ans</u>: Compare with $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

$$\lim_{n\to\infty} \frac{1+n\ln n}{n^2+5} \Big/ \frac{\ln n}{n} = 1$$

Since $\frac{\ln n}{n} > \frac{1}{n}$

for all $n \geq 3$

$\therefore \sum_{n=1}^{\infty} \frac{\ln n}{n}$ div. $\therefore$ <u>div</u>