Chap 10 Sequences and Series

Why do we care about infinite sequences and series?

<u>Example</u>

* $\int_0^1 (1+x^3)^{\frac{1}{2}}\,dx = ?$

* Solve

$$y'' - 2ty' - 2y = 0$$

Possible solution method:

* Given $f(x)$, can we find $a_n \in \mathbb{R}$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \ ?$$

* It is known that, if a_n exists, then

$$a_n = \frac{f^{(n)}(0)}{n!}$$

* Is it true that

$$f(x) \overset{?}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

for $x \neq 0$?

* Is it true that

$$f'(x) \overset{?}{=} \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\int_0^x f(t)\,dt \overset{?}{=} \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

Fundamental Questions

Sequence: $a_0, a_1, a_2, \cdots, a_n \cdots$

How to define

$$\lim_{n \to \infty} a_n = L \; ?$$

$$\boxed{\begin{array}{l}\text{Given any } \varepsilon > 0, \text{ there} \\ \text{exists an integer } N, \\ \text{such that } "n > N \Rightarrow |a_n - L| < \varepsilon" \\ (\text{for all } n > N, \text{ we have } |a_n - L| < \varepsilon)\end{array}}$$

How to define

$$\lim_{n\to\infty} a_n = \infty$$

$$-\infty \ ?$$

Given any $M \in \mathbb{R}$ there
$m \in \mathbb{R}$.
exists an integer N,
such that for all $n > N$,
we have $a_n > M$
$< m$

Example 1 $a_n = \{1, -1, 1, -1, \cdots\}$

$$\lim_{n\to\infty} a_n \begin{array}{l} \neq \pm\infty \\ \neq 1 \\ \neq -1 \\ \neq \text{any } L \in \mathbb{R} \end{array}$$

Example 2: $a_n = \{1, 0, 2, 0, 3, 0 \cdots\}$

$$\lim_{n\to\infty} a_n \neq \begin{array}{l} \pm\infty \\ \text{any } L \in \mathbb{R} \end{array}$$

Example 3: $a_n = \{1, 4, 9, 16, \cdots\}$

Then $\lim_{n\to\infty} a_n = \infty$

Since given any $M \in \mathbb{R}$

(assume $M > 0$)

take $N = [\sqrt{M}] + 1$,

for all $n > N$,

we have $a_n = n^2 > M$

Example 4: (Sandwich Thm)

$$\lim_{n\to\infty} \frac{\cos n}{n} = ?$$

Since $\frac{-1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$

and $\lim_{n\to\infty} \frac{-1}{n} = 0 = \lim_{n\to\infty} \frac{1}{n}$

$\therefore \lim_{n\to\infty} \frac{\cos n}{n} = 0$

Example 5: $\lim_{n\to\infty} \frac{\ln n}{n} = ?$

Ans: find $\lim_{\substack{x\to\infty \\ x\in\mathbb{R}}} \frac{\ln x}{x}$

l'Hôpital Rule:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

Example 6: $\lim_{n\to\infty}\left(\frac{n+1}{n-1}\right)^{n}$

$$= \lim_{n\to\infty}\left(e^{\ln\left(\frac{n+1}{n-1}\right)}\right)^{n}$$

$$= \lim_{n\to\infty} e^{n\ln\left(\frac{n+1}{n-1}\right)}$$

$$\lim_{x\to\infty} x\ln\left(\frac{x+1}{x-1}\right) = ?$$

$$x \ln\left(\frac{x+1}{x-1}\right) = x \ln\left(1+\frac{2}{x-1}\right)$$

$$= \frac{\ln\left(1+\frac{2}{x-1}\right)}{\frac{1}{x}}$$

$$\therefore \lim_{x\to\infty} = \frac{\ln\left(1+\frac{2}{x-1}\right)'}{\left(\frac{1}{x}\right)'}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{2}{x-1}} \cdot 2\left(\frac{-1}{(x-1)^2}\right)}{-\frac{1}{x^2}} = 2$$

$$\therefore \lim_{n\to\infty}\left(\frac{n+1}{n-1}\right)^n = e^{\lim\limits_{n\to\infty} n\ln\left(\frac{n+1}{n-1}\right)} = e^2$$

Example 7: $\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}}$

$$= \lim_{n\to\infty} \left(e^{\ln n}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} e^{\left(\frac{\ln n}{n}\right)}$$

$$= e^{\lim_{n\to\infty}\left(\frac{\ln n}{n}\right)}$$

$$= e^{0} = 1$$

Example 8: $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n,\ x\in\mathbb{R}$

$$= \lim_{n\to\infty} e^{\underline{n \ln\left(1+\frac{x}{n}\right)}}$$

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\ln\left(1+\frac{x}{n}\right)'}{\left(\frac{1}{n}\right)'}$$

$$\left(' = \frac{d}{dn}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{1}{1+\frac{x}{n}}\cdot\left(\frac{-x}{n^2}\right)}{\frac{-1}{n^2}} = x$$

$$= e^{\lim_{n\to\infty}(\cdots)} = e^x$$

Theorem (from advanced Calculus)

If there exists an integer N, such that

$a_n \leq a_{n+1}$ and $a_n \leq M$

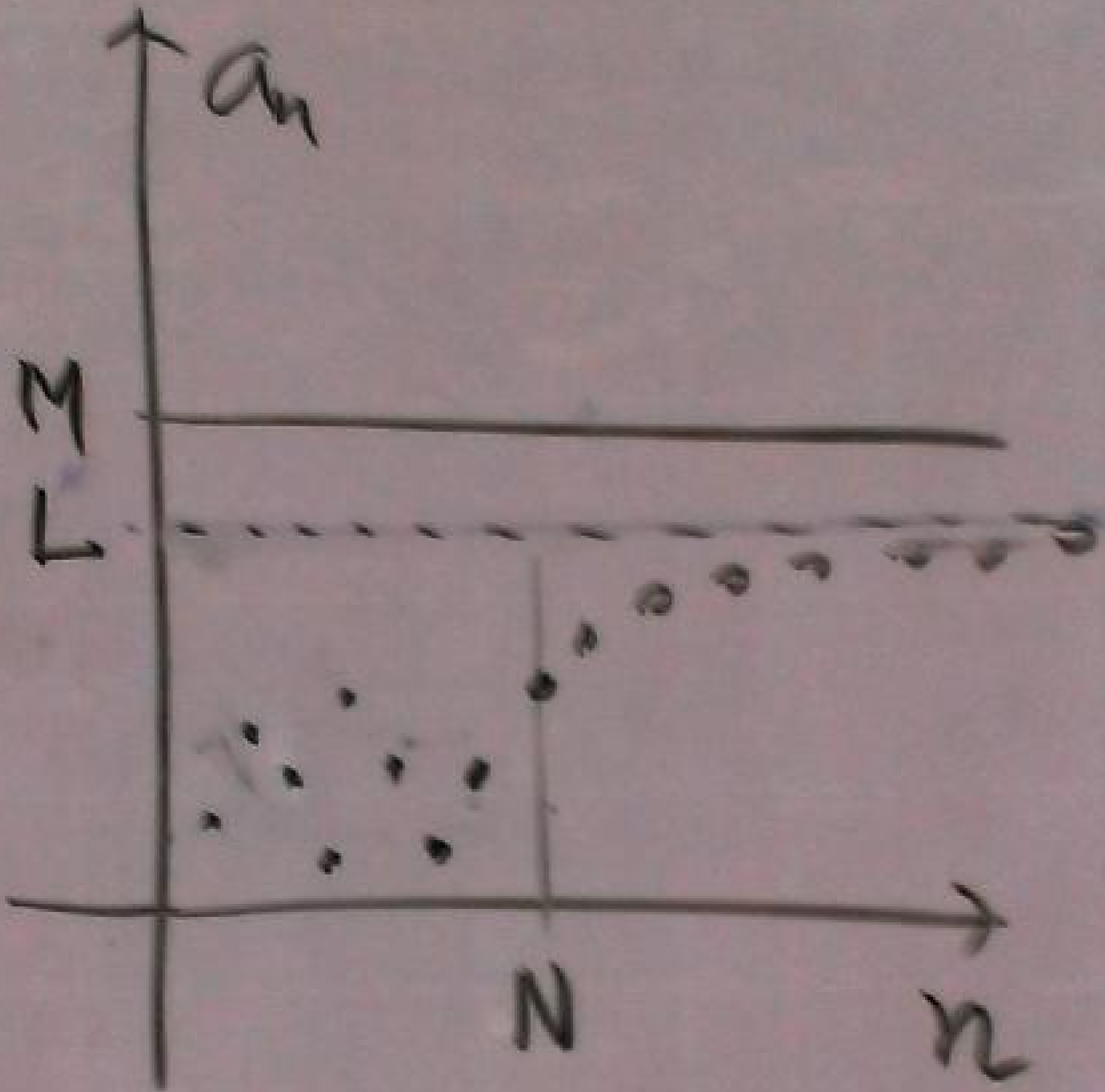
$a_n \geq a_{n+1}$ and $a_n \geq m$

for all $n > N$,

then $\lim_{n \to \infty} a_n$ exists

(and $\leq M$)

$\geq m$



a_n
M
L
N
n