How to check differentiability?

__Example__ $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

(a) Is f_x, f_y cont. at $(0,0)$?

(b) Is f differentiable at $(0,0)$?

__Note__ (a) true $\Longrightarrow$ (b) true

(a) false $\overset{?}{\Longrightarrow}$ (b) false

__For (a).__ $f_x(0,0) = \lim\limits_{h \to 0} \dfrac{f(h,0)}{h} = 0$

$$f_x(x,y) = \frac{(\ x^4,\ x^3y,\ xy^3,\ y^4)}{(x^2+y^2)^2} \quad (x,y) \neq (0,0)$$

direct inspection $\Rightarrow f_x$ not cont at $(0,0)$

(b) $\Delta f = f(x,y) - f(0,0) = f(x,y)$

$\Delta x = x - 0\ ,\ \Delta y = y - 0$

(*1): $\Delta f \stackrel{?}{=} f_x(0,0)\Delta x + f_y(0,0)\Delta y$

$+ \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

$\left(\text{or } + \varepsilon \sqrt{(\Delta x)^2 + (\Delta y)^2}\right)$

$\lim\limits_{(x,y)\to(0,0)} \varepsilon_1, \varepsilon_2, \varepsilon = 0$

$f_x(0,0) = 0,\ f_y(0,0) = 0$

$\Delta f \stackrel{?}{=} \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

$\frac{x^2 y}{x^2 + y^2} \stackrel{?}{=} \varepsilon_1 x + \varepsilon_2 y$

Take $x = y \to 0$

$$x \stackrel{?}{=} (\varepsilon_1 + \varepsilon_2)x$$

or $$1 \stackrel{?}{=} (\varepsilon_1 + \varepsilon_2) \to \underline{\underline{\text{No}}}$$

Another way to check:

$$D_{\vec{u}} f(0,0) \stackrel{?}{=} \left(f_x(0,0), f_y(0,0)\right) \cdot \vec{u}$$

where $\vec{u} = (\cos\theta . \sin\theta)$

$$D_{\vec{u}} f(0,0) = \lim_{r \to 0} \frac{f(ru_1, ru_2) - f(0,0)}{r}$$

$$= \cos^2\theta \sin\theta$$

$$\begin{pmatrix} f_x(0,0) = 0 \\ f_y(0,0) = 0 \end{pmatrix} \neq (0,0) \cdot \vec{u}$$

Remark on tangency

Then

Def $z = f(x,y)$
and $z = L(x,y)$ are
tangent at (x_0, y_0):

if $$\lim_{(x,y)\to(x_0,y_0)} \frac{|f(x,y)-L(x,y)|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0 \quad (*2)$$

~~Note: (*2) is exactly the same as (*1) on page 2~~

If $L(x,y) = A(x-x_0)+B(y-y_0)+C$
is tangent to $z=f(x,y)$ at (x_0,y_0)

$$\lim_{(x,y)\to(x_0,y_0)} \frac{|f(x,y)-A(x-x_0)-B(y-y_0)-C|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$$

Then (i) $\lim$ numerator $= 0$
(分子)

$\Rightarrow C = -f(x_0, y_0)$

(ii) $$\lim_{(x,y)\to(x_0,y_0)} \frac{\left|\left(f(x,y)-f(x_0,y_0)\right)-A(x-x_0)-B(y-y_0)\right|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$$

$$0 = \lim_{\substack{y=y_0\\ x\to x_0}} = \left|f_x(x_0,y_0)-A\right|$$

$$0 = \lim_{x=x_0,\, y\to y_0} = \left|f_y(x_0,y_0)-B\right|$$

$$\Rightarrow A = f_x(x_0,y_0),\ B = f_y(x_0,y_0),\ C = f(x_0,y_0)$$

This gives the only candidate of the tangent plane if it exists.
Note that, the tangent plane may or may not exist.