Brief answers to Midterm 2

May 16, 2016

Show all details. Problem 1-8: 12 pts each. Problem 9-10: 10 pts each. Total 116 points. **(averege score = 58.2 pts)**

1. True or False? Prove it if true. Give a counter example if false.

   If $f(x,y)$ is differentiable at $(0,0)$, then $f(x,y)$ is continuous at $(0,0)$.

   Remark: partial credits for at least writing correct definitions.

   Answer. (averege score = 3.81/12 pts: 31.8%)

   True. **(2 pts)**

   Definitions: see p774 and p790 (equivalent definitions also acceptable). **(5 pts)**

   proof:

   Since f is differentiable at $(0,0)$, both $f_x(0,0)$ and $f_y(0,0)$ exist and

   $$f(x,y) - f(0,0) = f_x(0,0)x + f_y(0,0)y + \epsilon_1 x + \epsilon_2 y$$

   where $\epsilon_1, \epsilon_2 \to 0$ as $(x,y) \to (0,0)$

   $\Rightarrow |f(x,y) - f(0,0)| = |f_x(0,0)||x| + |f_y(0,0)||y| + |\epsilon_1||x| + |\epsilon_2||y| \to 0$ as $(x,y) \to (0,0)$.

   therefore f is continuous at $(0,0)$ **(5 pts)**.

2. Evaluate

   $$\frac{d}{dy}\int_1^{2+y^2} \frac{\cos(xy)}{x}\,dx.$$

   Answer. (averege score = 3.25/12 pts: 27.1%)

   Let $G(y,z) = \int_1^z \frac{\cos(xy)}{x}\,dx$. We need to compute

   $$\frac{d}{dy}G(y,2+y^2) = G_y(y,2+y^2) + G_z(y,2+y^2)\cdot 2y \ (\mathbf{8pts})$$

   $$= -\int_1^{2+y^2} \sin(xy)\,dx + \frac{\cos((2+y^2)y)}{2+y^2}\cdot 2y \ \mathbf{(2\ pts)}$$

   $$= \frac{1}{y}\big(\cos((2+y^2)y) - \cos y\big) + \frac{\cos((2+y^2)y)}{2+y^2}\cdot 2y.\ (\mathbf{2pts})$$

   Answer only without derivation: either 0 pts or 12 pts.

3. Find the equation of plane normal to the following curve at $(1,-1,1)$

$$\begin{cases} x^2+2y^2+3z^2=6 \\ x+y+z=1 \end{cases}$$

Answer. (averege score = 8.38/12 pts: 69.8%)

First compute the two gradients at $(1,-1,1)$

$$(2x,4y,6z)_{(1,-1,1)}=(2,-4,6)\ \textbf{(2 pts)}$$

and

$$(1,1,1)|_{(1,-1,1)}=(1,1,1).\ \textbf{(2 pts)}$$

The normal vector of the plane is parallel to the outer product of these two gradients:

$$n=\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}=(5,-2,-3).\ \textbf{(4 pts)}$$

Therefore, the equation of the plane is

$$5(x-1)-2(y+1)-3(z-1)=0.\ \textbf{(4 pts)}$$

4. Show that, for any $a\in\mathbb{R}$, the point $(x_0,y_0)=(0,0)$ is a critical point of the function

$$f_a(x,y)=(a-1)(x+y)^2+(a+1)(x-2y)^2.$$

For what values of a is the point $(0,0)$ a local minimum, a local maximum and a saddle point, respectively?

Answer. (averege score = 8.62/12 pts: 71.8%)

Method.1 For $a\geq 1$, $f_a(x,y)=(a-1)(x+y)^2+(a+1)(x-2y)^2$ is a sum of two squares. And $(0,0)$ admits the minimum 0. **(4 pts)**
For $-1<a<1$, $f_a(x,y)=(a+1)(x-2y)^2-(1-a)(x+y)^2$ is a difference of two squares. And $(0,0)$ is a saddle point since $f_a(1,-1)=9(a+1)>0$ and $f_a(2,1)=-9(1-a)<0$. **(4 pts)**
For $a\leq -1$, $f_a(x,y)=-[(1-a)(x+y)^2+(-1-a)(x-2y)^2]$ is the negative of sum of two squares. And $(0,0)$ admits the maximum 0. **(4 pts)**

Method.2 First compute the gradient at $(0,0)$

$$\nabla f_a(0,0)=(2(a-1)(x+y)+2(a+1)(x-2y),\ 2(a-1)(x+y)+2(a+1)(x-2y)(-2))|_{(0,0)}=(0,0),$$

and thus $(0,0)$ is a critical point. **(3 pts)**

Now compute the second derivatives

$$(f_a)_{xx}(0,0)=2(a-1)+2(a+1)=4a,$$

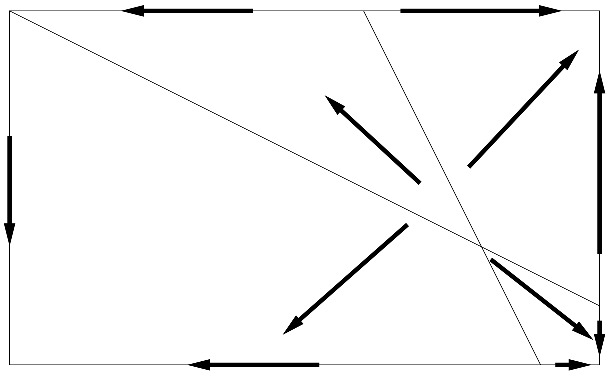

Figure 1: The gradient analysis for problem 5.

$$(f_a)_{yy}(0,0) = 2(a-1) + 8(a+1) = 10a + 6 = 2(5a+3),$$

and

$$(f_a)_{xy}(0,0) = 2(a-1) - 4(a+1) = -2(a+3).$$

A local minimum occurs as $(f_a)_{xx} > 0$ and $(f_a)_{xx}(f_a)_{yy} - (f_a)_{xy}^2 > 0$, i.e.

$$\begin{aligned} & a > 0,\ 8a(5a+3) - 4(a+3)^2 = 36(a^2-1) > 0 \\ \Leftrightarrow\quad & a > 1.\ \textbf{(3 pts)} \end{aligned}$$

A local maximum occurs as $(f_a)_{xx} < 0$ and $(f_a)_{xx}(f_a)_{yy} - (f_a)_{xy}^2 > 0$, i.e.

$$\begin{aligned} & a > 0,\ 8a(5a+3) - 4(a+3)^2 = 36(a^2-1) > 0 \\ \Leftrightarrow\quad & a < -1.\ \textbf{(3 pts)} \end{aligned}$$

A saddle point occurs as $(f_a)_{xx}(f_a)_{yy} - (f_a)_{xy}^2 < 0$, i.e.

$$\begin{aligned} & 8a(5a+3) - 4(a+3)^2 = 36(a^2-1) < 0 \\ \Leftrightarrow\quad & -1 < a < 1.\ \textbf{(3 pts)} \end{aligned}$$

For $a = 1$, $f_a(x,y) = 2(x-2y)^2$. And thus $(0,0)$ admits a local minimum.
For $a = -1$, $f_a(x,y) = -2(x+y)^2$. And this $(0,0)$ admits a local maximum.

Getting the two equalities $a = \pm 1$ correctly: **(extra 2 pts)**.

5. Find absolute maxima and minima of $f(x,y) = x^2 + xy + y^2 - 6x$ on the rectangular domain $0 \le x \le 5$, $-3 \le y \le 0$.

   Answer. (averege score = 9.18/12 pts: 76.4%)

First find the gradient:

$$\nabla f(x,y) = (2x + y - 6, x + 2y).\ \textbf{(2 pts)}$$

Therefore one can plot the gradients as in figure 1. From the plot it is easy to see that $f(4,-2)=-12$ is the only local minimum since there is no local minimum on the boundary **(3 pts)**. Therefore $f(4,-2)=-12$ is also the absolute minimum **(2 pts)**.

Moreover, the local maximum consists of the three corners $(0,-3)$, $(5,-3)$ and $(5,0)$. Upon comparing the values of f on the corners $(0,-3)$, $(5,-3)$ and $(5,0)$ it follows that the absolute maxima is $f(0,-3)=9$ **(5 pts)**.

6. Use the method of Lagrangian multiplier (this method only) to find the point closest to the origin on the curve of intersection of the plane $2y+4z=5$ and the cone $z^2=4x^2+4y^2$.

   Answer. (averege score = 6.93/12 pts: 57.8%)

   Minimize $f(x,y,z)=x^2+y^2+z^2$, subject to the constraints

   $$g_1(x,y,z)=2y+4z-5=0, \quad g_2(x,y,z)=4x^2+4y^2-z^2=0. \ \textbf{(3 pts)}$$

   $$\begin{cases} \nabla f=\lambda_1\nabla g_1+\lambda_2\nabla g_2 \\ g_1(x,y,z)=0 \\ g_2(x,y,z)=0. \end{cases} \Rightarrow \begin{cases} 2x= \qquad \lambda_2 8x \\ 2y=\lambda_1 2+\lambda_2 8y \\ 2z=\lambda_1 4+\lambda_2(-2z) \\ 2y+4z-5=0 \\ 4x^2+4y^2-z^2=0. \end{cases} \ \textbf{(3 pts)}$$

   case 1:

   $\lambda_2=1/4$, no solution.

   case 2:

   $x=0$, $(y,z)=(1/2,1)$ or $(-5/6,5/3)$. Compare values of f on these two points: minimum occurs when $(x,y,z)=(0,1/2,1)$. Shortest distance $=\sqrt{5}/2$ **(6 pts)**.

7. Derive Taylor's formula for $f(x,y)$, assuming all partial derivatives of any order of f are continuous. Find a quadratic approximation of $f(x,y)=\ln(2x+y+1)$ near the origin.

   Answer. (averege score = 5.01/12 pts: 41.7%)

   Derivation: page 840 of the textbook **(6 pts)**.

   Quadratic approximation:

   $$f_x(x,y)=\frac{2}{1+2x+y}, \qquad f_y(x,y)=\frac{1}{1+2x+y}, \ \textbf{(1 pts)}$$

   $$f_{xx}(x,y)=\frac{-4}{(1+2x+y)^2}, \quad f_{xy}(x,y)=\frac{-2}{(1+2x+y)^2}, \quad f_{yy}(x,y)=\frac{-1}{(1+2x+y)^2} \ \textbf{(1 pts)}$$

   $$\begin{aligned} Q(x,y) &= f(0,0)+f_x(0,0)x+f_y(0,0)y+\frac{1}{2}(f_{xx}(0,0)x^2+2f_{xy}(0,0)xy+f_{yy}(0,0)y^2) \\ &= 0+2x+y-\frac{1}{2}(4x^2+4xy+y^2) \ \textbf{(4 pts)} \end{aligned}$$

8. Evaluate

$$\int_0^2 \int_y^2 x^2 \cos(xy)\ dxdy$$

Answer. (averege score = 7.32/12 pts: 61.0%)

$$\begin{aligned}
\int_0^2 \int_y^2 x^2 \cos(xy)\ dxdy &= \int_0^2 \int_0^x x^2 \cos(xy)\ dydx \ \textbf{(6 pts)} \\
&= \int_0^2 x\sin(xy)\Big|_0^x\ dx \\
&= \int_0^2 x\sin(x^2)\ dx \ \textbf{(3 pts)} \\
&= -\frac{1}{2}\cos(x^2)\Big|_0^2 = -\frac{1}{2}(\cos 4 - 1). \ \textbf{(3 pts)}
\end{aligned}$$

9. Find the Taylor series generated by $\sin^{-1} x$, centered at 0.

Hint: what is $\frac{d}{dx}\sin^{-1} x$?

Answer. (averege score = 3.83/10 pts: 38.3%)

$$\sin^{-1} x = \int_0^x (1-t^2)^{\frac{-1}{2}} dt \ (\textbf{2pts}) =$$

$$= \int_0^x (1 - \frac{1}{2}(-t^2) + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}(-t^2)^2 - \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}(-t^2)^3 + \cdots + + (-1)^n \frac{\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-1}{2}}{n!}(-t^2)^n + \cdots)dt \ (\textbf{5pts})$$

$$= x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^7}{7} + \cdots + \frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}\cdot\frac{x^{2n+1}}{2n+1} + \cdots \ (\textbf{3pts})$$

10. Evaluate $\sum_{n=0}^{\infty} \frac{x^n}{n+2}$ on $|x| < 1$ using computational rules of power series.

Answer. (averege score = 1.98/10 pts: 19.8%)

The value is 1/2 as $x = 0$. **(2 pts)**. For $x \neq 0$,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{x^n}{n+2} &= x^{-2}\left(\sum_{n=0}^{\infty} \frac{x^{n+2}}{n+2}\right) \\
&= x^{-2}\left(\sum_{n=0}^{\infty} \int_0^x t^{n+1}\, dt\right) \ \textbf{(3 pts)} \\
&= x^{-2}\left(\int_0^x \sum_{n=0}^{\infty} t^{n+1}\, dt\right) \\
&= x^{-2}\left(\int_0^x \frac{t}{1-t}\, dt\right) \ \textbf{(3 pts)} \\
&= x^{-2}\left(-x - \ln(1-x)\right) = -\frac{1}{x} - \frac{\ln(1-x)}{x^2}. \ \textbf{(2 pts)}
\end{aligned}$$