Midterm 1

April 02, 2019

Show all details.

1. (6 pts) Give formal definition of $\lim_{n\to\infty} a_n = L$.

   Answer. See page 552 of the textbook.

2. (6+8 pts) Show that $\sum_{k=1}^{\infty} k^{-2}$ converges and evaluate $\lim_{n\to\infty} \frac{\log\left(\sum_{k=n}^{\infty} k^{-2}\right)}{\log n}$.

   Hint: Give an estimate of the form $f(n) \leq \sum_{k=n}^{\infty} k^{-2} \leq g(n)$ and use it to find the limit.

   Answer.

$$\because\ f(k) = k^{-2}\ \geq 0\ \searrow\ \text{and}\ \int_1^{\infty} \frac{1}{x^2}\,dx \text{ converges. } \therefore \text{ Converges. } (\mathbf{6pts})$$

$\because$

$$\frac{1}{n} = \int_n^{\infty} \frac{1}{x^2}\,dx \leq \sum_{k=n}^{\infty} \frac{1}{k^2} \leq \int_{n-1}^{\infty} \frac{1}{x^2}\,dx = \frac{1}{n-1}$$

$$\Rightarrow -1 \leq \frac{\log\left(\sum_{k=n}^{\infty} k^{-2}\right)}{\log n} \leq \frac{-\log(n-1)}{\log n}$$

And,

$$\lim_{n\to\infty} \frac{-\log(n-1)}{\log n} = -1.$$

$\therefore$ the limit $= -1$. **(8 pts)**

3. (18 pts) Are the following series convergent? Explain.

   (a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+n}+n}$ (b) $\sum_{n=1}^{\infty} \frac{3^n n! n!}{(2n)!}$ (c) $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^{\frac{n}{2}}}$

   Answer.

   (a) Use limit comparison theorem, compare it with $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$. Convergent.

   (b) Ratio test: $\lim \frac{a_{n+1}}{a_n} = \frac{3}{4} < 1$. Convergent.

   (c) Root test: $\lim \sqrt[n]{a_n} = \lim \frac{\sqrt[n]{n}}{\sqrt{\ln n}} = \frac{1}{\infty} = 0 < 1$. Convergent.

4. (8 pts) Find $\sum_{n=1}^{\infty} nx^n$ and $\sum_{n=1}^{\infty} n^2x^n$ on $|x|<1$ using computational rules for power series (multiplication, differentiation, integration, etc.). Need NOT prove your equality holds.

Answer.

$$
\begin{aligned}
& 1+x+x^2+\ldots = \frac{1}{1-x} \ \textbf{(2 pts)} \\
\Rightarrow \quad & x+2x^2+3x^3+\ldots = x\left(\frac{1}{1-x}\right)' = \frac{x}{(1-x)^2} \ \textbf{(3 pts)} \\
\Rightarrow \quad & x+2^2x^2+3^2x^3+\ldots = x\left(\frac{x}{(1-x)^2}\right)' = \frac{x(1+x)}{(1-x)^3} \ \textbf{(3 pts)}.
\end{aligned}
$$

5. (a) (6 pts) Show that $1-\frac{1}{2\cdot 1!}+\frac{1}{4\cdot 2!}-\cdots+(-1)^n\frac{1}{2^n\cdot n!}+\cdots$ converges absolutely.

   (b) (4+6 pts) Find the sum of the series in (a) and prove your answer (show the equality holds).

Answer.

(a) Ratio test:

$$\lim_{n\to\infty}\frac{\frac{1}{2^{n+1}(n+1)!}}{\frac{1}{2^n n!}} \to 0<1, \quad \text{convergent. } \textbf{(6 pts)}$$

(b)

$$e^{-1/2}. \ \textbf{(4 pts)}$$

$$|R_n(-\frac{1}{2})| \le \frac{e^{c_{n+1}}}{(n+1)!}\frac{1}{2^{n+1}} \to 0 \text{ where } c_{n+1}\in(-\frac{1}{2},0)\ . \ \textbf{(6 pts)}$$

6. (8 pts) Find the Taylor series generated by $\sin^{-1}x$, centered at 0.

Answer.

$$\sin^{-1}x = \int_0^x (1-t^2)^{\frac{-1}{2}}dt =$$

$$= \int_0^x (1-\frac{1}{2}(-t^2)+\frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}(-t^2)^2-\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}(-t^2)^3+\cdots++(-1)^n\frac{\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-1}{2}}{n!}(-t^2)^n+\cdots)dt$$

$$= x+\frac{1}{2}\frac{x^3}{3}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^7}{7}+\cdots+\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}\cdot\frac{x^{2n+1}}{2n+1}+\cdots$$

7. (8 pts) Give an approximation of $\int_0^1 \sin(x^3)dx$ to within 10^{-5}. Give the formula of the approximation (need not find the numerical value) and explain why the error is less than 10^{-5}.

Answer.

$$\int_0^1 \sin(x^3)dx = \int_0^1 \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x^3)^{2n-1}}{(2n-1)!}dx = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\int_0^1 (x^3)^{2n-1}dx}{(2n-1)!}$$

(3 pts)

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(6n-2)(2n-1)!} = a_1 - a_2 + a_3 - a_4 + \cdots$$

Since

$$a_4 = \frac{1}{22 \cdot 7!} = \frac{1}{110880} \approx 0.9 \times 10^{-5}.$$

(2 pts) From error estimate for alternating series, the approximation $a_1 - a_2 + a_3$ will do the job with error less than $a_4 < 10^{-5}$ **(3 pts)**.

8. (8 pts) Use any method to evaluate $\lim_{x \to 0} \dfrac{\frac{\sin x}{x} - \cos x}{x^2}$.

Answer.

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

(1 pts)

$$\text{answer} = \lim_{x \to 0} \frac{(1 - \frac{1}{3!}x^2 + \cdots) - (1 - \frac{1}{2!}x^2 + \cdots)}{x^2} = \frac{1}{3}$$

(7 pts)

9. True or False? Prove it if true, give a counter example if false.

   (a) (4 pts) If $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = -1$, then it converges at $x = 1$.

   (b) (4 pts) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} n a_n$ converges.

   (c) (6 pts) If $g(x) = f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ on $|x| < 1$, then $f(x) = g(x)$ on $|x| < 1$.

Answer.

(a) False **(1 pts)**. $\sum_n \frac{x^{`n}}{n}$ converges at $x = -1$, but diverges at $x = 1$**(3 pts)**.

(b) False **(1 pts)**. $\sum_n \frac{(-1)^n}{n}$ converges, but $\sum_n (-1)^n$ diverges **(3 pts)**.

(c) False **(2 pts)**. Take $f(x) = e^{-1/x^2}$ for $x \neq 0$, and $f(x) = 0$ for $x = 0$. Then $f^{(n)}(0) = 0$, $\forall\, n \geq 0$ **(3 pts)**. Therefore $f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0 \neq f(x)$, if $x \neq 0$ **(1 pts)**.