Brief Solution to Final Exam

Jun 13, 2019

Show all details.

1. (10 pts) Change
$$\int_{\sqrt{2}}^{2}\int_{\sqrt{4-y^2}}^{y} dxdy$$
into an equivalent polar integral. Write down the integral. Need not evaluate it.

Answer.

$x = \sqrt{4-y^2}$ corrsponds to $r = 2$.

$y = 2$ corrsponds to $r = 2\csc\theta$.

$x = y$ corrsponds to $\theta = \frac{\pi}{4}$.

$$\text{Answer } = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{2}^{2\csc\theta} rdrd\theta$$

2. (10 pts) Find the volume in the first octant bounded by the three coordinate planes and the surface $z = 4 - x^2 - y$.

Answer.

$$\int_0^2\int_0^{4-x^2}\int_0^{4-x^2-z} dy\, dz\, dx$$

3. (10 pts) Replace
$$\int_0^{2\pi}\int_0^1\int_0^{\sqrt{2-r^2}} rdz\, dr\, d\theta$$
by triple integrals in spherical coordinates in the order $d\rho d\phi d\theta$. Write down the triple integral. Need not evaluate it.

Hint: start by drawing the cross section $\{\theta = 0\} \cap$ (this domain) in the $r - z$ plane. This domain is obtained by rotating the cross section around the z axis.

Answer.

$$\int_0^{2\pi}\left(\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2}} \rho^2\sin\phi\, d\rho d\phi + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\csc\phi} \rho^2\sin\phi\, d\rho d\phi\right) d\theta$$

4. (10 pts) Let $D = \{1 \le x \le 2,\ 0 \le xy \le 1,\ 0 \le z \le 1\}$. Evaluate

$$\iiint_D (x^2y + xyz)dV.$$

Answer.

Let $u = x, v = xy, w = z$.

Therefore $x = u, y = v/u, z = w, \quad x^2y + xyz = uv + vw. \quad J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{u}$

$$\text{Answer} \ = \int_0^1 \int_0^1 \int_1^2 \frac{uv + vw}{u}\, du\, dv\, dw = \frac{1}{2} + \frac{\ln 2}{4}$$

5. (12 pts) True or false? Give details.
If $f(x, y, z)$ has continuous first derivatives in a domain D, and $C = \{(x(t), y(t), z(t)),\ 0 \le t \le 1\}$ is a smooth curve in D. Then $\int_C \nabla f \cdot d\boldsymbol{r}$ depends only on f, $(x(0), y(0), z(0))$ and $(x(1), y(1), z(1))$.
Answer.

$$\begin{aligned} & \int_C \nabla f \cdot \mathbf{T}\, ds \\ = & \int_0^1 (f_x(x(t), y(t), z(t))x'(t) + f_y(x(t), y(t), z(t))y'(t) + f_z(x(t), y(t), z(t))z'(t))\, dt \\ = & \int_0^1 \partial_t(f(x(t), y(t), z(t))\, dt = f(x(1), y(1), z(1)) - f(x(0), y(0), z(0)). \end{aligned}$$

6. (24 pts) Let $R = \{x^2 + y^2 > 1/4\}$, $\boldsymbol{F}(x, y) = (2y, x)$, $\boldsymbol{G}(x, y) = (\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}})$, $\boldsymbol{H}(x, y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$. Which one(s) of $\boldsymbol{F}$, $\boldsymbol{G}$ and $\boldsymbol{H}$ is (are) conservative on R? (That is, which one(s) of $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$, $\int_C \boldsymbol{G} \cdot d\boldsymbol{r}$ and $\int_C \boldsymbol{H} \cdot d\boldsymbol{r}$ is (are) zero on <u>every</u> closed loop C in R?) Explain.
Answer.
$\boldsymbol{F}$ is not conservative since $\partial_y(2y) = 2 \ne 1 = \partial_x(x)$
$\boldsymbol{H}$ is not conservative since for some $r(t) = (a\cos t, a\sin t)$, $0 \le t \le 2\pi$,

$$\oint_C \boldsymbol{H} \cdot \boldsymbol{T}\, ds = \int_0^{2\pi} \left(-\frac{\sin t}{a}, \frac{\cos t}{a}\right) \cdot (-a\sin t, a\cos t)\, dt = \int_0^{2\pi} 1\, dt = 2\pi \ne 0.$$

$\boldsymbol{G}$ is conservative. There are two methods to check it.

Method.1 It suffices to check whether $\oint_C \boldsymbol{G}\cdot\boldsymbol{T}\,ds = 0$ for any loop of the form $C = \{x^2+y^2 = a^2,\ 1/2 \le a\}$ since $\boldsymbol{G}$ satisfies the component test.

$$\partial_y\left(\frac{x}{\sqrt{x^2+y^2}}\right) = -\frac{xy}{(x^2+y^2)^{3/2}} = \partial_x\left(\frac{y}{\sqrt{x^2+y^2}}\right).$$

Let $r(t) = (a\cos t, a\sin t)$, $0 \le t \le 2\pi$.

$$\oint_C \boldsymbol{G}\cdot\boldsymbol{T}\,ds = \int_0^{2\pi} (\cos t, \sin t)\cdot(-a\sin t, a\cos t)\,dt = \int_0^{2\pi} 0\,dt = 0.$$

Method.2 Find $f = f(x,y)$ such that $\boldsymbol{G} = \nabla f$.

$$\begin{aligned}
& f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}} \\
\Rightarrow\quad & f(x,y) = \sqrt{x^2+y^2} + g(y) \\
\Rightarrow\quad & f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}} + g'(y) = \frac{y}{\sqrt{x^2+y^2}} \\
\Rightarrow\quad & g(y) = C \\
\Rightarrow\quad & f(x,y) = \sqrt{x^2+y^2} + C.
\end{aligned}$$

7. Let $\boldsymbol{F}(x,y) = (M(x,y), N(x,y))$ have continuous first and second derivatives everywhere in $\mathbb{R}^2$ and let $R = \{x^2+y^2 < 1, x > 0, y > 0\}$.

   (a) (8 pts) State Green's Theorem in both forms for $\boldsymbol{F}$ on R.

   (b) (16 pts) Take $\boldsymbol{F} = (y, x)$ and verify both forms on R. That is, evaluate both line integral and double integral and check that they are the same. Do this for both forms.

Answer.

(a)

$$\oint_C \boldsymbol{F}\cdot\mathbf{n}ds = \oint_C Mdy - Ndx = \iint_R (M_x + N_y)\,dxdy$$
$$\oint_C \boldsymbol{F}\cdot\mathbf{T}ds = \oint_C Mdx + Ndy = \iint_R (N_x - M_y)\,dxdy$$

where C is a piecewise smooth, simple closed curve that encloses the region R.

(b) Let $r_1(t) = (\cos t, \sin t)$, $0 \le t \le \frac{\pi}{2}$; $r_2(t) = (0, 1-t)$, $0 \le t \le 1$; $r_3(t) = (t, 0)$, $0 \le t \le 1$

. Then

$$\begin{aligned}
\oint_C \boldsymbol{F}\cdot\mathbf{n}\,ds &= \int_0^{\frac{\pi}{2}} (\sin t\cos t - \cos t(-\sin t))\,dt + \int_0^1 ((1-t)\cdot(-1) - 0\cdot 0)\,dt \\
&+ \int_0^1 (0\cdot 0 - t\cdot 1)\,dt \\
&= 1 - \frac{1}{2} - \frac{1}{2} = 0 = \iint_R (0+0)\,dxdy = \iint_R (M_x + N_y)\,dxdy, \\
\oint_C \boldsymbol{F}\cdot\mathbf{T}\,ds &= \int_0^{\frac{\pi}{2}} (\sin t(-\sin t) + \cos t\cos t)\,dt + \int_0^1 ((1-t)\cdot 0 + 0\cdot(-1))\,dt \\
&+ \int_0^1 (0\cdot 1 + t\cdot 0)\,dt \\
&= 0+0+0 = \iint_R (1-1)\,dxdy = \iint_R (N_x - M_y)\,dxdy.
\end{aligned}$$

8. (8 pts) Write down the definitions of '$f(x,y)$ is continuous at $(0,0)$' and '$f(x,y)$ is differentiable at $(0,0)$', respectively.

   Answer. See solution for midterm 02.

9. (10 pts) Evaluate

$$\frac{d}{dy}\int_1^{2+y^2} \frac{\cos(xy)}{x}dx.$$

   Answer. See solution for midterm 02.

10. (12 pts) Find the sum of the series $1 - \frac{1}{2\cdot 1!} + \frac{1}{4\cdot 2!} - \cdots + (-1)^n \frac{1}{2^n\cdot n!} + \cdots$ and prove that the equality holds.

   Answer. See solution for midterm 01.