# Brief solutions to quiz 02

1. Suppose that $f$ is infinitely many times differentiable on $R$. State Taylor's Theorem for $f$ and write down the formula for $R_n(x)$ (in any form you choose).

   **Ans**: See page 607 of the textbook. Both formula are correct.

2. Find the Taylor series generated by $\sin x$ at $x = 0$. For what values of $x$ does $\sin x$ equal this series? Explain.

   **Ans**:

   (a) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$.

   (b) The series converges for all $x \in (-\infty, \infty)$.

   (c) Since for all $x \in (-\infty, \infty)$,

   $$\lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)!}\right|}{\left|\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right|} = \lim_{n\to\infty} \frac{|x|^2}{(2n+2)(2n+3)} = 0 < 1,$$

   by the Ratio Test, the series converges for all $x \in (-\infty, \infty)$.

3. Find the sum of the series

   $$\frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \cdots + (-1)^{n-1}\frac{1}{n2^n} + \cdots$$

   **Ans**:

   Replace $\frac{1}{2^n}$ by $x^n$. Then take differentiate it to eliminate $\frac{1}{n}$ term. This will result in a series that you recognize. Then you can integrate once to get the original series and then evaluate at $x = 1/2$. Answer: $\ln(1 + \frac{1}{2})$.

4. Let $f(x) = \exp(\frac{-1}{x^2})$ for $x \neq 0$ and $f(0) = 0$. Is $f$ differentiable everywhere? Explain.

   **Ans**:

   (a) Yes, $f$ is differentiable everywhere.

   (b) For $c \neq 0$, since $g(y) = e^y$ is differentiable at $-\frac{1}{c^2}$ and $h(x) = -\frac{1}{x^2}$ is differentiable at $c$, by the Chain Rule, $e^{-1/x^2}$ is differentiable at $c$, and

   $$f'(c) = \frac{2\,e^{-\frac{1}{c^2}}}{c^3}.$$

(c) Since

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{e^{-\frac{1}{h^2}}-0}{h}=\lim_{y\to+\infty}\frac{e^{-y^2}}{\frac{1}{y}}=\lim_{y\to+\infty}\frac{y}{e^{y^2}}=0,$$

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{e^{-\frac{1}{h^2}}-0}{h}=\lim_{y\to-\infty}\frac{e^{-y^2}}{\frac{1}{y}}=\lim_{y\to-\infty}\frac{y}{e^{y^2}}=0,$$

$f$ is differentiable at $x=0$, and $f'(0)=0$.

5. Find the area of the surface generated by evolving $x=\cos t$, $y=2+\sin t$, $0\le t\le 2\pi$ about the $x$-axis.

**Ans**:

$$\int_0^{2\pi} 2\pi(2+\sin t)\sqrt{x'(t)^2+y'(t)^2}dt=\int_0^{2\pi} 2\pi(2+\sin t)dt=8\pi^2 \tag{1}$$