Brief Solutions to Final Exam

Show all details.

1. (15+1 pts) Replace

$$\int_0^{2\pi}\int_0^1\int_0^{\sqrt{2-r^2}} rdz\ dr\ d\theta$$

by triple integrals in spherical coordinates in the order $d\rho d\phi d\theta$ and $d\phi d\rho d\theta$, respectively. Need not evaluate them.

Ans: $d\rho d\phi d\theta$:

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2}} \rho^2\sin\phi\ d\rho\ d\phi\ d\theta + \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\csc\phi} \rho^2\sin\phi\ d\rho\ d\phi\ d\theta \quad \text{(8 pts)}$$

$d\phi d\rho d\theta$:

$$\int_0^{2\pi}\int_0^{1}\int_0^{\frac{\pi}{2}} \rho^2\sin\phi\ d\phi\ d\rho\ d\theta + \int_0^{2\pi}\int_1^{\sqrt{2}}\int_0^{\sin^{-1}\frac{1}{\rho}} \rho^2\sin\phi\ d\phi\ d\rho\ d\theta \quad \text{(8 pts)}$$

2. (15 pts) Let $D = \{1 \le x \le 2,\ 0 \le xy \le 1,\ 0 \le z \le 1\}$. Evaluate

$$\int\int\int_D (x^2y + xyz)dV.$$

Ans: Let $u = x$, $v = xy$, $w = z$. Then $x = u$, $y = v/u$, $z = w$. We have $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \dfrac{1}{u}$.

$$\text{Ans} = \int_0^1\int_0^1\int_1^2 v + \frac{vw}{u}\ du\ dv\ dw = \frac{1}{2} + \frac{1}{4}\ln 2.$$

3. Let $\boldsymbol{F}(x,y) = (M(x,y), N(x,y))$ have continuous first and second derivatives everywhere in $\mathbb{R}^2$ and let $R = \{x^2+y^2 < 1, x > 0, y > 0\}$.

   (a) (10 pts) State Green's Theorem in both forms for $\boldsymbol{F}$ on R.

   (b) Prove either one of the two forms on R (20 pts).
   (10 pts) If you can not prove it for general $\boldsymbol{F}$, do the following: Take $\boldsymbol{F} = (y,x)$ and verify both forms on R. That is, evaluate both line integral and double integral and check that they are the same. Do this for both forms.

Ans:

(a) Let C be the boundary of R.

$$\oint_C \boldsymbol{F}\cdot\boldsymbol{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy,$$

$$\oint_C \boldsymbol{F}\cdot\boldsymbol{T}\,ds = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

(b1) *(Prove of Green's Theorem, for instance, in the first form.)* Let $C = C_1 \cup C_2 \cup C_3$, where

$$
\begin{aligned}
C_1: &\quad \boldsymbol{r}(t) = \left(\sqrt{1-t^2}, t\right), \quad && 0 \le t \le 1, \\
C_2: &\quad \boldsymbol{r}(t) = (0, 1-t), && 0 \le t \le 1, \\
C_3: &\quad \boldsymbol{r}(t) = (t, 0), && 0 \le t \le 1.
\end{aligned}
$$

Then (changing of variables $u = 1-t$, $v = \sqrt{1-t^2}$)

$$
\begin{aligned}
\int_{C_1} M\,dy &= \int_0^1 M(\sqrt{1-t^2}, t)\cdot 1\,dt = \int_0^1 M(\sqrt{1-t^2}, t)\,dt, \\
\int_{C_2} M\,dy &= \int_0^1 M(0, 1-t)\cdot(-1)\,dt = -\int_0^1 M(0,u)\,du, \\
\int_{C_3} M\,dy &= \int_0^1 M(t,0)\cdot 0\,dt = 0, \\
\int_{C_1} N\,dx &= \int_0^1 N(\sqrt{1-t^2}, t)\cdot \frac{d\sqrt{1-t^2}}{dt}\,dt = -\int_0^1 N(v, \sqrt{1-v^2})\,dv, \\
\int_{C_2} N\,dx &= \int_0^1 N(0, 1-t)\cdot 0\,dt = 0, \\
\int_{C_3} N\,dx &= \int_0^1 N(t,0)\cdot 1\,dt = \int_0^1 N(t,0)\,dt.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int_C M\,dy - N\,dx \\
&= \left(\int_{C_1} M\,dy + \int_{C_2} M\,dy + \int_{C_3} M\,dy\right) - \left(\int_{C_1} N\,dx + \int_{C_2} N\,dx + \int_{C_3} N\,dx\right) \\
&= \int_0^1 M(\sqrt{1-t^2}, t)\,dt - \int_0^1 M(0,u)\,du + \int_0^1 N(v, \sqrt{1-v^2})\,dv - \int_0^1 N(t,0)\,dt.
\end{aligned}
$$

On the other hand,

$$\begin{aligned}
&\iint_R \left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right) dx\,dy \\
&= \iint_R \frac{\partial M}{\partial x}\,dx\,dy + \iint_R \frac{\partial N}{\partial y}\,dx\,dy \\
&= \int_0^1 \int_0^{\sqrt{1-y^2}} \frac{\partial M}{\partial x}\,dx\,dy + \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\partial N}{\partial y}\,dy\,dx \\
&= \int_0^1 \Big[M(x,y)\Big]_{x=0}^{x=\sqrt{1-y^2}} dy + \int_0^1 \Big[N(x,y)\Big]_{y=0}^{y=\sqrt{1-x^2}} dx \\
&= \int_0^1 \Big(M(\sqrt{1-y^2},y)-M(0,y)\Big)\,dy + \int_0^1 \Big(N(x,\sqrt{1-x^2})-N(x,0)\Big)\,dx.
\end{aligned}$$

Hence,

$$\int_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right) dx\,dy.$$

(b2) *(Verifying the case $\boldsymbol{F}=(y,x)$.)* Let $C=C_1\cup C_2\cup C_3$, where

$$\begin{aligned}
C_1:&\quad \boldsymbol{r}(t)=(\cos t,\sin t)\,, && 0\le t\le \tfrac{\pi}{2},\\
C_2:&\quad \boldsymbol{r}(t)=(0,1-t)\,, && 0\le t\le 1,\\
C_3:&\quad \boldsymbol{r}(t)=(t,0)\,, && 0\le t\le 1.
\end{aligned}$$

We have

$$\oint_C y\,dy - x\,dx = \int_0^{\frac{\pi}{2}} \sin t\cos t+\cos t\sin t\,dt + \int_0^1 (1-t)\,(-1)\;dt + \int_0^1 (-t)\;dt = 0,$$

$$\iint_R \left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right) dx\,dy = \iint_R 0\,dx\,dy = 0,$$

and

$$\oint_C x\,dx + y\,dy = \int_0^{\frac{\pi}{2}} \cos t\,(-\sin t)+\sin t\cos t\,dt + \int_0^1 0\,dt + \int_0^1 0\,dt = 0,$$

$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\,dy = \iint_R 0\,dx\,dy = 0.$$

4. (15 pts) True or false? Give details.

   If $f(x,y,z)$ has continuous first derivatives in a domain D, and $C=\{(x(t),y(t),z(t)),\ 0\le t\le 1\}$ be a smooth curve in D. Then $\int_C \nabla f\cdot \boldsymbol{T}ds$ depends only on f, $(x(0),y(0),z(0))$ and $(x(1),y(1),z(1))$.

   Ans: True.

Let $g(t) = f(x(t), y(t), z(t))$. Then

$$\frac{dg}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \nabla f \cdot \frac{\partial \boldsymbol{r}}{\partial t}.$$

Hence

$$\int_C \nabla f \cdot \boldsymbol{T}\, ds = \int_C \nabla f \cdot d\boldsymbol{r} = \int_0^1 \nabla f \cdot \frac{d\boldsymbol{r}}{dt}\, dt = \int_0^1 \frac{dg}{dt}\, dt = \Big[g(t)\Big]_{t=0}^{t=1}$$
$$= g(1) - g(0) = f(x(1), y(1), z(1)) - f(x(0), y(0), z(0)).$$

5. (30 pts) Let $R = \{1/4 \le x^2 + y^2 \le 4\}$, $\boldsymbol{F}(x, y) = (2y, x)$, $\boldsymbol{G}(x, y) = (x, y)$, $\boldsymbol{H}(x, y) = (\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2})$.
Which one(s) of $\boldsymbol{F}$, $\boldsymbol{G}$ and $\boldsymbol{H}$ is (are) conservative on R? (That is, which one(s) of $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$, $\int_C \boldsymbol{G} \cdot d\boldsymbol{r}$ and $\int_C \boldsymbol{H} \cdot d\boldsymbol{r}$ is (are) zero on every closed loop C in R?) Explain.
Ans: (4% for answer and 6% for reason each)

(a) Since $2 = \frac{\partial}{\partial y}(2y) \ne \frac{\partial}{\partial x}(x) = 1$. Hence, $\boldsymbol{F}$ is NOT conservative.
(b) Since $\boldsymbol{G} = \nabla g$, where $g = \frac{1}{2}(x^2 + y^2)$. Hence, $\boldsymbol{G}$ is conservative.
(c) Let $C : \boldsymbol{r}(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$. Then

$$\int_C \boldsymbol{H} \cdot d\boldsymbol{r} = \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\; dt = 2\pi.$$

Hence, $\boldsymbol{H}$ is NOT conservative.

6. (15 pts) Let $\boldsymbol{F}(x, y, z) = z\boldsymbol{k}$ and $S = \{x^2 + y^2 + z^2 = 4,\ x > 0,\ y > 0,\ z > 0\}$. Evaluate $\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\; d\sigma$ where $\boldsymbol{n}$ is unit normal of S pointing away from the origin.
Ans: We parametrize the surface S by

$$\boldsymbol{r}(\theta, \varphi) = (2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi)\,, \quad 0 \le \theta \le \frac{\pi}{2},\ 0 \le \varphi \le \frac{\pi}{2}. \qquad (3\%)$$

Then $\boldsymbol{r}_\theta = (-2\sin\varphi\sin\theta, 2\sin\varphi\cos\theta, 0)$, $\boldsymbol{r}_\varphi = (2\cos\varphi\cos\theta, 2\cos\varphi\sin\theta, -2\sin\varphi)$, and

$$\boldsymbol{r}_\theta \times \boldsymbol{r}_\varphi = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -2\sin\varphi\sin\theta & 2\sin\varphi\cos\theta & 0 \\ 2\cos\varphi\cos\theta & 2\cos\varphi\sin\theta & -2\sin\varphi \end{vmatrix}$$
$$= \left(-4\sin^2\varphi\cos\theta, -4\sin^2\varphi\sin\theta, -4\sin\varphi\cos\varphi\right). \qquad (6\%)$$

Take

$$\boldsymbol{n} = -\frac{\boldsymbol{r}_\theta \times \boldsymbol{r}_\varphi}{\|\boldsymbol{r}_\theta \times \boldsymbol{r}_\varphi\|} = \frac{\left(4\sin^2\varphi\cos\theta, 4\sin^2\varphi\sin\theta, 4\sin\varphi\cos\varphi\right)}{\|\boldsymbol{r}_\theta \times \boldsymbol{r}_\varphi\|}. \qquad (1\%)$$

and we have $d\sigma = \|\boldsymbol{r}_\theta \times \boldsymbol{r}_\varphi\| \; d\theta \, d\varphi$,

$$\boldsymbol{F} \cdot \boldsymbol{n} = (0, 0, 2\cos\varphi) \cdot \frac{\left(4\sin^2\varphi\cos\theta, 4\sin^2\varphi\sin\theta, 4\sin\varphi\cos\varphi\right)}{\|\boldsymbol{r}_\theta \times \boldsymbol{r}_\varphi\|} = \frac{8\sin\varphi\cos^2\varphi}{\|\boldsymbol{r}_\theta \times \boldsymbol{r}_\varphi\|}.$$

Hence,

$$\iint\limits_S \boldsymbol{F} \cdot \boldsymbol{n} \, d\sigma = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{8\sin\varphi\cos^2\varphi}{\|\boldsymbol{r}_\theta \times \boldsymbol{r}_\varphi\|} \cdot \|\boldsymbol{r}_\theta \times \boldsymbol{r}_\varphi\| \; d\theta \, d\varphi$$

$$= 8\int_0^{\frac{\pi}{2}} \cos^2\varphi \sin\varphi \, d\varphi \int_0^{\frac{\pi}{2}} d\theta = 8\left[-\frac{1}{3}\cos^3\varphi\right]_{\varphi=0}^{\varphi=\frac{\pi}{2}} \Big[\theta\Big]_{\theta=0}^{\theta=\frac{\pi}{2}} = \frac{4\pi}{3}. \quad (5\%)$$