Final Exam

June 10, 2004, 10:10 AM. SHOW ALL YOUR WORK.

1. (5+5 pts) The derivative of $f(x, y, z)$ at a point $P = (1, 0, 0)$ is greatest in the direction of $(1, -1, 1)/\sqrt{3}$. In this direction the value of the derivative is 2
   (a) What is the derivative in the direction of $(1, 1, -1)/\sqrt{3}$?
   (b) Find the tangent plane of the level surface of f at P.

2. (10+5 pts)
   (a) State (need not prove) the second derivative test for $f(x, y)$. You may assume that f and all its first and second derivatives are continuous in an open region R.
   (b) Give an example of a function $f(x, y)$ with $(1, 1)$ as a saddle point.

3. (5+10 pts) Use the method of Lagrangian multipliers (and NOT any other methods) to solve the following problems.
   (a) Find the nearest points from origin on the curve $x^2 + xy + 2y^2 = 1$.
   (b) Find the extreme values of the function $f(x, y, z) = xy + z^2$ on the circle in which the plane $y - 2x = 0$ intersects the sphere $x^2 + y^2 + z^2 = 1$.

4. (5+5 pts) Switch the order of integration of
$$\int_0^1 \int_{-1}^0 \int_0^{y^2} dzdydx$$
to $dydxdz$ and $dxdydz$ respectively. You don't need to find the numerical value of the integral.

5. (10 pts) Compute the Jacobian for the stretched polar coordinate $\frac{\partial(x,y)}{\partial(\rho,\theta)}$, where $x = \rho^{1/3}\cos\theta$ and $y = \rho^{1/3}\sin\theta$.

6. (6+10 pts)
   (a) State the Green's Theorem both in Normal and Tangential Forms. You may assume M, N and all their first derivatives are continuous in a region R and its boundary C.
   (b) Verify both of them on the region $R = \{x^2 + y^2/4 \leq 1\}$ with $M(x, y) = x$, $N(x, y) = -2y$. In other words, you have to evaluate the line integral and double integral independently for the Normal form, verify they are the same and repeat the process for the Tangential form.

7. (10 pts) Verify by direct computation that the curl of a conservative vector field is identically zero provided all the components of the vector field and their first derivatives are continuous.

8. (15 pts) Compute

$$\int\int_{x^2+y^2+z^2=1,\, z>0} z d\sigma \tag{1}$$

You can EITHER compute the surface integral directly OR apply the divergence Theorem on the hemisphere $\{x^2+y^2+z^2 < 1,\, z > 0\}$ with the observation that $\boldsymbol{n} = (x, y, z)$ on the surface in (1).