Midterm 1

Mar 28, 2016

Show all details.

1. (16 pts) Are the integrals $\int_0^1 \frac{1}{\sqrt{x+x^3}}dx$ and $\int_1^\infty \tan(\frac{1}{x})dx$ convergent? Explain.

2. (6 pts) Give formal definition of $\lim_{n\to\infty} a_n = L$.

3. (8 pts) Evaluate $\lim_{n\to\infty} \left(\frac{n-1}{n+1}\right)^n$.

4. (6+10 pts) Show that $\sum_{k=1}^{\infty} k^{-2}$ converges and evaluate* $\lim_{n\to\infty} \frac{\log\left(\sum_{k=n}^{\infty} k^{-2}\right)}{\log n}$. Give details.

   (If the limit is p, this means that $\sum_{k=n}^{\infty} k^{-2}$ is approximately n^p. Find p and prove it.)

   Hint: recall the proof of one of the convergence tests.

5. (8 pts) State (need not prove) Taylor's Theorem (or Taylor's formula). Assume the function f has derivatives of all orders on $\mathbb{R}$. Write down n terms approximation and the formula of the remainder term.

6. (6+12 pts)

   (a) Show that the series $1 - \frac{1}{2\cdot 1!} + \frac{1}{4\cdot 2!} - \cdots + (-1)^n \frac{1}{2^n \cdot n!} + \cdots$ converges *absolutely*.

   (b) Find the sum of the series in (a). <u>Prove your answer</u> (that is, the equality holds).

7. (8 pts) Find $\sum_{n=1}^{\infty} nx^n$ and $\sum_{n=1}^{\infty} n^2 x^n$ on $|x| < 1$ using computational rules of power series. Need NOT prove your equality holds as in previous question.

8. (6 pts) Find a power series that converges on $[1, 5)$ and diverges elsewhere. Explain.

9. (12 pts) Give an approximation of $\int_0^{\frac{1}{2}} \sin(x^2)dx$ to within 10^{-8}. Give the formula of the approximation, but need not find the numerical value. Explain why the error is less than 10^{-8}.

10. (8+10+10 pts) True or False? Prove it if true, give a counter example if false.

    (a) If $\sum a_n$ converges, then $\sum na_n$ converges.

    (b) * If $\sum a_n x^n$ converges on $|x| < 1$, then $\sum \sqrt{n} a_n x^n$ also converges on $|x| < 1$.

    (c) If $g(x) = f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ on $|x| < 1$, then $f(x) = g(x)$ on $|x| < 1$.