Brief Solutions to Midterm 2

May 08, 2014

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1. (10%) Evaluate
$$\frac{d}{dy}\int_1^{2+y^2}\frac{\cos(xy)}{x}dx$$

Ans:
$$\begin{aligned}&= \frac{\cos(xy)}{x}\Big|_{x=2+y^2} + \int_1^{2+y^2}\frac{d}{dy}\frac{\cos(xy)}{x}dx \\ &= \frac{\cos((2+y^2)y)}{2+y^2} + \int_1^{2+y^2} -\sin(xy)dx \quad (8\%) \\ &= \frac{\cos((2+y^2)y)}{2+y^2} + \frac{1}{y}\Big(\cos((2+y^2)y) - \cos y\Big) \quad (2\%)\end{aligned}$$

2. (10%) True or false? Prove it or give a counter example.

Assume $f(x,y)$, $f_x(x,y)$ and $f_y(x,y)$ are all continuous in R^2. Let $C = \{(x,y),\ f(x,y) = f(0,0)\}$ and $\mathcal{T}$ be a tangent vector of C at $(0,0)$. Then $\nabla f(0,0)\cdot\mathcal{T} = 0$.

Ans: True (3%).

Proof: C is a curve. We may assume it is parametrized by $(x(t),y(t))$ with $(x(0),y(0)) = (0,0)$. Therefore $\mathcal{T} = (x'(0),y'(0))$. Since f is constant on C, we have
$$0 = \frac{d}{dt}f(x(t),y(t))|_{t=0} = \nabla f(0,0)\cdot\mathcal{T} \quad (7\%).$$

3. (10%) Use Lagrangian multipliers (and only Lagrangian multipliers) to find extreme values of $f(x,y,z) = xy + 2z^2$ on
$$\begin{cases} x^2+y^2+z^2 = 9 \\ x - y = 0\end{cases}$$

Ans: Let $g_1(x,y,z) = x^2+y^2+z^2-9$, $g_2(x,y,z) = x-y$.

Solve from
$$\begin{cases} g_1(x,y,z) = 0, \\ g_2(x,y,z) = 0, \\ \nabla f(x,y,z) = \lambda_1\nabla g_1(x,y,z) + \lambda_2\nabla g_2(x,y,z)\end{cases} \quad (6\%).$$

Max: $(x,y,z) = (0,0,\pm 3)$. $f_{\max} = 18$, (2%).

Min: $(x,y,z) = \pm(\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2},0)$. $f_{\min} = \frac{9}{2}$, (2%).

4. (10%) Find the absolute maximum and minimum of $f(x,y) = 2 + 2x + 2y - x^2 - y^2$ in the region bounded by $x = 0$, $y = 0$ and $x + y = 6$.

Ans:

(a) *Interior points.*

$$\begin{cases} f_x(x,y) = 2 - 2x = 0 \\ f_y(x,y) = 2 - 2y = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 1 \end{cases}$$

The critical point of f is $(1,1)$. $f(1,1) = 4$. (2%)

(b) *Boundary points.*

(i) Line segment $(0,0)$–$(0,6)$ $(x = 0)$.

$$f(0,y) = 2 + 2y - y^2, \quad 0 \le y \le 6.$$

$\frac{d}{dy} f(0,y) = 2 - 2y = 0$ implies $y = 1$. $f(0,1) = 3$. (1%)

(ii) Line segment $(0,0)$–$(6,0)$ $(y = 0)$.

$$f(x,0) = 2 + 2x - x^2, \quad 0 \le x \le 6.$$

$\frac{d}{dx} f(x,0) = 2 - 2x = 0$ implies $x = 1$. $f(1,0) = 3$. (1%)

(iii) Line segment $(0,6)$–$(6,0)$ $(y = 6 - x)$.

$$f(x,6-x) = 2 + 2x + 2(6-x) - x^2 - (6-x)^2 = -2x^2 + 12x - 22, \quad 0 \le x \le 6.$$

$\frac{d}{dx} f(x,6-x) = -4x + 12 = 0$ implies $x = 3$. $f(3,3) = -4$. (1%)

(iv) Endpoints of above three line segents.
$f(0,0) = 2$, $f(0,6) = -22$, $f(6,0) = -22$. (3%)

Therefore,

$$\text{absolute maximum} = f(1,1) = 4,$$
$$\text{absolute mimimum} = f(0,6) = f(6,0) = -22. \quad (2\%)$$

5. (10%) Let $f(x,y) = x^3 + y^3$ and $g(r,\theta) = f(r\cos\theta, r\sin\theta)$. Evaluate $\partial_r^2 g + (\partial_r g)/r + (\partial_\theta^2 g)/r^2$

Ans:

$$\frac{\partial g}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = 3x^2 \cdot \cos\theta + 3y^2 \sin\theta = 3r^2\cos^3\theta + 3r^2\sin^3\theta, \quad (2\%)$$

$$\frac{\partial^2 g}{\partial r^2} = \frac{\partial}{\partial r}(\frac{\partial g}{\partial r}) = 6r\cos^3\theta + 6r\sin^3\theta, \quad (2\%)$$

$$\frac{\partial g}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -3r^3\cos^2\theta\sin\theta + 3r^3\cos\theta\sin^2\theta, \quad (2\%)$$

$$\frac{\partial^2 g}{\partial \theta^2} = \frac{\partial}{\partial \theta}(\frac{\partial g}{\partial \theta}) = 3r^3(3\sin^2\theta\cos\theta + 3\cos^2\theta\sin\theta - \cos\theta - \sin\theta). \quad (2\%)$$

Therefore,

$$\begin{aligned}\frac{\partial^2 g}{\partial \theta^2}+\frac{1}{r}\frac{\partial g}{\partial r}+\frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} &= (6r\cos^3\theta+6r\sin^3\theta)+(3r\cos^3\theta+3r\sin^3\theta)\\ &\quad +3r(3\sin^2\theta\cos\theta+3\cos^2\theta\sin\theta-\cos\theta-\sin\theta)\\ &= 6r\cos\theta+6r\sin\theta. \qquad (2\%)\end{aligned}$$

6. (10%) Derive the Taylor expansion of $f(x,y,z)$ around $(0,0,0)$ up to quadratic terms of x, y and z and an expression of the remainder term, R_2. You may assume that f and all its first and second derivatives are all continuous in R^3.

Ans: Assume that f and all its first and second derivatives are continuous on R^3. Parametrize the line segment joining $O(0,0,0)$ to $P(h,k,\ell)$ by

$$\begin{cases} x = th \\ y = tk \\ z = t\ell \end{cases}, \quad 0 \le t \le 1.$$

Let $F(t) = f(th, tk, t\ell)$. Then (2%)

$$\begin{aligned}
F'(t) &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial t} = hf_x+kf_y+\ell f_z, \qquad (1\%)\\
F''(t) &= h(hf_{xx}+kf_{yy}+\ell f_{xz})+k(hf_{yx}+kf_{yy}+\ell f_{yz})+\ell(hf_{zx}+kf_{zy}+\ell f_{zz})\\
&= h^2f_{xx}+k^2f_{yy}+\ell^2 f_{zz}+2hkf_{xy}+2k\ell f_{yz}+2h\ell f_{xz}, \qquad (2\%)\\
F'''(t) &= h^2(hf_{xxx}+kf_{xxy}+\ell f_{xxz})+k^2(hf_{yyx}+kf_{yyy}+\ell f_{yyz})\\
&\quad +\ell^2(hf_{zzx}+kf_{zzy}+\ell f_{zzz})+2hk(hf_{xyz}+kf_{xyy}+\ell f_{xyz})\\
&\quad +2k\ell(hf_{yzx}+kf_{yzy}+\ell f_{yzz})+2h\ell(hf_{xzx}+kf_{xzy}+\ell f_{xzz})\\
&= h^3f_{xxx}+k^3f_{yyy}+\ell^3f_{zzz}+3h^2kf_{xxy}+3h^2\ell f_{xxz}\\
&\quad +3hk^2f_{xyy}+3k^2\ell f_{yyz}+3h\ell^2 f_{xzz}+3k\ell^2 f_{yzz}+6hk\ell f_{xyz}, \qquad (2\%)
\end{aligned}$$

where all the partial derivatives of f are evaluated at $(th, tk, t\ell)$. By Taylor's theorem, there exists $c\in(0,1)$ such that

$$\begin{aligned}
F(1) &= F(0)+F'(0)(1-0)+\frac{1}{2!}F''(0)(1-0)^2+\frac{1}{3!}F'''(c)(1-0)^3,\\
f(h,k,\ell) &= \Big(f+(hf_x+kf_y+\ell f_z)\\
&\quad +\frac{1}{2!}(h^2f_{xx}+k^2f_{yy}+\ell^2f_{zz}+2hkf_{xy}+2k\ell f_{yz}+2h\ell f_{xz})\Big)\Big|_{(0,0,0)}\\
&\quad +\Big(\frac{1}{3!}(h^3f_{xxx}+k^3f_{yyy}+\ell^3f_{zzz}+3h^2kf_{xxy}+3h^2\ell f_{xxz}\\
&\quad +3hk^2f_{xyy}+3k^2\ell f_{yyz}+3h\ell^2f_{xzz}+3k\ell^2f_{yzz}+6hk\ell f_{xyz})\Big)\Big|_{(ch,ck,c\ell)}. \qquad (1\%)
\end{aligned}$$

Hence, the Taylor expansion of $f(x,y,z)$ at $(0,0)$ up to second order is

$$\begin{aligned} f(x,y,z) = f(0,0,0) &+ (xf_x + yf_y + zf_z) \\ &+ \frac{1}{2!}(x^2 f_{xx} + y^2 f_{yy} + z^2 f_{zz} + 2xyf_{xy} + 2yzf_{yz} + 2xzf_{xz}) \\ &+ \frac{1}{3!}(x^3 f_{xxx} + y^3 f_{yyy} + z^3 f_{zzz} + 3x^2 y f_{xxy} + 3x^2 z f_{xxz} \\ &\quad + 3xy^2 f_{xyy} + 3y^2 z f_{yyz} + 3xz^2 f_{xzz} + 3yz^2 f_{yzz} + 6xyz f_{xyz}), \end{aligned}$$

where all the first and second derivatives are evaluated at $(0,0,0)$, and all the third derivatives are evaluated at (cx, cy, cz) for some $c \in (0,1)$. (2%)

7. (10%) Evaluate $(\frac{\partial U}{\partial P})_V$ and $(\frac{\partial U}{\partial T})_V$ at $(P,V,T) = (1,2,2)$ where $U(P,V,T) = T\exp(-P/V)$ with the constraint $PV = T$.

   Ans:

   we have

$$(\frac{\partial U}{\partial P})_V(1,2,2) = \frac{\partial}{\partial P}U(P,V,T(P,V))_{(1,2,2)} = \left(\frac{\partial U}{\partial P} + V\frac{\partial U}{\partial T}\right)_{(1,2,2)} \quad (4\%)$$

$$= -\exp(\frac{-1}{2}) + 2\exp(\frac{-1}{2}) = \exp(\frac{-1}{2}) \quad (1\%)$$

$$(\frac{\partial U}{\partial T})_V(1,2,2) = \frac{\partial}{\partial T}U(P(V,T),V,T)_{(1,2,2)} = \left(\frac{1}{V}\frac{\partial U}{\partial P} + \frac{\partial U}{\partial T}\right)_{(1,2,2)} \quad (4\%)$$

$$= -(\frac{1}{2})\exp(\frac{-1}{2}) + \exp(\frac{-1}{2}) = \frac{1}{2}\exp(\frac{-1}{2}) \quad (1\%)$$

8. (10%) Evaluate $\int_0^2 \int_y^2 \frac{\sin x}{x} dxdy$.

   Ans: we have

$$\begin{aligned} &\int_0^2 \int_y^2 \frac{\sin(x)}{x} dxdy \\ &= \int_0^2 \int_0^x \frac{\sin(x)}{x} dydx \quad (6\%) \\ &= \int_0^2 (\frac{\sin(x)}{x})(x) dx \quad (2\%) \\ &= \int_0^2 \sin(x) dx = 1 - \cos(2) \quad (2\%) \end{aligned}$$

9. (20%) Let $f(x,y) = \dfrac{x^3 + y^3}{x^2 + y^2}$, for $(x,y) \neq (0,0)$ and $f(0,0) = 0$. $P = (0,0)$ and $\boldsymbol{u}^\theta = (\cos\theta, \sin\theta)$, $\theta \in [0, 2\pi]$.

(a) Is f continuous at $(0,0)$? Explain.

(b) For fixed θ, write down the definition of the directional derivative $\left(\frac{df}{ds}\right)_{\boldsymbol{u}^{\theta},P}$ and evaluate it.

(c) Does f have a linear approximation at $(0,0)$? Explain.

Ans:

(a) (6%) Let

$$x = r\cos(u), y = r\sin(u)$$

then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} \frac{r^3((\cos(u))^3 + (\sin(u))^3)}{r^2} = 0$$

then f is continuous at 0.

(b) (7%)

$$\left(\frac{df}{ds}\right)_{\boldsymbol{u}^{\theta},P} = \lim_{h\to 0} \frac{(f(h\cos(\theta), h\cos(\theta)) - f(0,0))}{h}$$

$$= \lim_{h\to 0} \frac{(h((\cos(\theta))^3 + \sin(\theta))^3)}{h} = (\cos(\theta))^3 + (\sin(\theta)))^3$$

(c) (7%) since $\nabla f(0,0)\cdot \boldsymbol{u}(\theta) = (1,1)\cdot(\cos(\theta), \sin(\theta)) \neq (\cos(\theta))^3 + \sin(\theta))^3)$
then f is not differentiable at $(0,0)$
Therefore f does not have linear approximation.