Brief solution to Midterm Exam 2

Dec 10, 2013, 10:10AM

1. (16 pts) Graph the function $y = \dfrac{x^3 + 2x - 2}{x + 1}$. Indicate all critical points and points of inflection.

   Sol:

   Note that it is easier to compute the derivatives with the following form:

   $$y = x^2 - x + 3 + \frac{-5}{x + 1}$$

   3% for y', 3% for y'', 2% for critcal points, 2% for critcal points of inflection, 6% for correct graph.

2. (10 pts) Let f be a real valued function defined on $\{x \geq 0\}$ satisfying

   (a): $f(0) = -1$,

   (b): $f'(x) \geq 1/2$ for all $x \geq 0$.

   Prove that $f(x) = 0$ has exactly one solution on $\{x \geq 0\}$.

   Sol:

   Existence: 5%. Use Intermediate Value Theorem.

   Uniquenss: 5%. Use Mean Value Theorem.

3. (16 pts) Find the limits of the following expressions:

   $$\text{(a)} \quad \lim_{x \to 0^+} x^x \qquad \text{(b)} \quad \lim_{x \to 0} \frac{x^2 \cos \frac{1}{x}}{\sin x}$$

   Sol:

   (a): 1. (b): $= \lim_{x \to 0} \frac{x}{\sin x}(x^2 \cos \frac{1}{x}) = 1 \cdot 0 = 0$.

4. (16 pts) State both parts of Fundamental Theorem of Calculus, prove that part 1 implies part 2, then evaluate

   $$\frac{d}{dx} \int_{\sin x}^{1} e^{t^2} dt.$$

   Sol:

   Ans $= -e^{\sin^2 x} \cdot \cos x$.

5. (16 pts) Evaluate

$$\text{(a)} \quad \int_1^2 \frac{1}{x(1+\ln^2 x)} dx \qquad \text{(b)} \quad \int_0^4 x\sqrt{2x+1}\, dx$$

Sol:

(a): $\tan^{-1}(\ln 2)$ (b): Let $y = \sqrt{2x+1}$. Ans $= \frac{298}{15}$.

6. (10 pts) Evaluate

$$\lim_{n\to\infty} \sum_{k=n}^{2n} \frac{n}{k^2}$$

Sol:

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=n}^{2n} \frac{n^2}{k^2} \ (4\%) \ = \int_1^2 \frac{1}{x^2} \ (3\%) \ = \frac{1}{2} \ (3\%)$$

7. (16 pts) Find the volume and surface area of the object obtained by rotating the region $\{(x-2)^2 + y^2 \le 1, \ x \ge 2\}$ around the y axis. Note the surface area consists of two parts, one generated by a half circle, the other generated by a line segment.

Sol: 4 % for formula, 4% for solution, both for V and A.

$$V = \int_2^3 (2\pi x)(2\sqrt{1-(x-2)^2})\ dx \text{ or } = \int_{-1}^1 \pi\big((2+\sqrt{1-y^2})^2 - 2^2\big)\ dy = \frac{4}{3}\pi + 2\pi^2$$

$$A = 8\pi + 2\int_2^3 (2\pi x)\sqrt{1+(\frac{dy}{dx})^2}\ dx \text{ or } = 8\pi + \int_{-1}^1 2\pi(2+\sqrt{1-y^2})\sqrt{1+(\frac{dx}{dy})^2}\ dy = 4\pi^2 + 12\pi$$