Brief solutions to Midterm Exam 1

1. (8 pts) Find $\lim\limits_{y\to+\infty} y\sin\dfrac{2}{\sqrt{y}}$.

   Ans:
   $$= \lim_{y\to+\infty} \sqrt{y}\; 2\; \frac{\sqrt{y}}{2} \sin\frac{2}{\sqrt{y}} = \infty\cdot 2\cdot 1 = \infty.$$

2. (12 pts) Give precise definition of $\lim_{x\to 0^+} \frac{1}{x} = \infty$ and show that it is true (using the $\varepsilon - \delta$ argument).

   Ans:

   For any $M \in R$, there exist $\delta > 0$ such that $0 < x < \delta$ implies $f(x) > M$.

   Proof: If $M > 0$, take $\delta = 1/M$. If $M \leq 0$, take any $\delta > 0$ (for example $\delta = 1$) will do. Then verify (details skipped).

3. (8 pts) Find dy/dx where $y = x^x, x > 0$. Need not simplify your expression.

   Ans: $= x^x(1 + \ln x)$.

4. (12 pts) Find y' and y'' at $(1, -1)$ where $y(x)$ is implicitly given by $\tan(x+y)+\sin(x^2+y) = 0$.

   Ans:

   $\frac{d}{dx}$ once:
   $$\big(\sec^2(x+y)\big)(1+y') + \big(\cos(x^2+y)\big)(2x+y') = 0,$$
   evaluate at $x = 1, y = -1$, one gets $y' = -3/2$.

   $\frac{d}{dx}$ twice:
   $$\big(2\sec^2(x+y)\tan(x+y)\big)(1+y')^2+\big(\sec^2(x+y)\big)y''-\big(\sin(x^2+y)\big)(2x+y')^2+\big(\cos(x^2+y)\big)(2+y'') = 0$$
   evaluate at $x = 1, y = -1, y' = -3/2$, one gets $y'' = -1$.

5. (12 pts) Find the smallest n such that $\dfrac{d^n}{dx^n}(x^{10}\sin x)|_{x=0}$ is nonzero and find this value.

   Ans: $n = 11$. $\dfrac{d^{11}}{dx^{11}}(x^{10}\sin x)|_{x=0} = 11!$.

6. (12 pts) True or False? If true, prove it. If false, give a counter example.

   If $|f(x) - (3x + 2)| \leq |x|^{1.5}$ for all $x \in R$, then f is differentiable at $x = 0$.

   Ans: True.

   Step 1: let $x = 0$, we have $f(0) = 2$.

   Step 2:
   $$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x) - (3x + 2)}{x - 0} + 3.$$

Since $|\frac{f(x)-(3x+2)}{x-0}| \le |x|^{0.5}$, it follows from Sandwich Theorem that

$$\lim_{x\to 0} \frac{f(x)-(3x+2)}{x-0} = 0.$$

Therefore

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 3, \quad \text{differentiable.}$$

7. (12 pts) Write down $L(x, x_0)$, the linear approximation of f near x_0. Find an approximate value of $\sin(\frac{\pi}{3} - 0.01)$ such that the error of the approximation is smaller than 5×10^{-5}. (Hint: choose x_0 carefully)

   Ans:

   $L(x, x_0) = f(x_0) + f'(x_0)(x - x_0)$. Choose $x_0 = \frac{\pi}{3}$. $\sin(\frac{\pi}{3} - 0.01) \approx \sin(\frac{\pi}{3}) + \cos\frac{\pi}{3}\,(-0.01) = \frac{\sqrt{3}}{2} - 0.005$.

$$|\text{Error}| \le \frac{1}{2}|\sin c|\; 0.01^2 \le 5 \times 10^{-5}$$

8. (12 pts) True or False? (prove it if true, correct it if false).

   Since $x \mapsto \ln x$ and $x \mapsto e^x$ are inverse function to each other and $\frac{d}{dx}\ln x = \frac{1}{x}$. Therefore $\frac{d}{dx}e^x = \frac{1}{\frac{d}{dx}\ln x} = \frac{1}{\frac{1}{x}} = x$.

   Ans: False.

   Correction: $\frac{d}{dx}e^x = \frac{1}{(\frac{d}{dy}\ln y)_{y=e^x}} = \frac{1}{(\frac{1}{y})_{y=e^x}} = e^x$.

9. (12 pts)

   Find absolute maximum and absolute minimum of $f(x) = x^{1/3}(x - 1/2)$ on $[-1, 1]$.

   Ans:

   $f'(x) = \frac{1}{6}x^{\frac{-2}{3}}(8x - 1)$.

   $x = 0$, f is not differentiable.

   $x = 1/8$, $f'(1/8) = 0$.

   End points: $x = -1, 1$.

   Compare $f(x)$ on $x = -1, 1, 0, 1/8$, we find that absolute maximum $= f(-1) = 3/2$, absolute minimum $= f(1/8) = -3/16$.