Brief solutions to Final Exam

1. (8 pts) Find the solutions for $\dfrac{dy}{dx} = 3x^2 e^{-y}$.

   Ans:
$$\int e^y\,dy = \int 3x^2\,dx \quad\Rightarrow\quad e^y = x^3 + C \quad\Rightarrow\quad y = \ln(x^3 + C).$$

2. (8 pts) Write down the form of partial fraction expansion for $\dfrac{x^7}{(1-x^4)^2}$. Need NOT find the undetermined coefficients.

   Ans:
$$\begin{aligned}\frac{x^4}{(1-x^4)^2} &= \frac{x^7}{(1+x)^2(1-x)^2(1+x^2)^2}\\ &= \frac{A}{1+x} + \frac{B}{(1+x)^2} + \frac{C}{1-x} + \frac{D}{(1-x)^2} + \frac{Ex+F}{1+x^2} + \frac{Gx+H}{(1+x^2)^2}.\end{aligned}$$

3. (8 pts) Order e^x, x^x, $(\ln x)^x$ and x^e from slowest to fastest growing rate as $x \to \infty$.

   Ans: As $x \to \infty$, e(constant) grows slower than $\ln x$, $\ln x$ grows slower than x. Hence e^x grows slower than $(\ln x)^x$, $(\ln x)^x$ grows slower than x^x. On the other hand, x^e grows slower than e^x. Thus, the answer (the order of growing rate from the slowest to the fastest) is
$$x^e, \quad e^x, \quad (\ln x)^x, \quad x^x.$$

4. (64 pts) Evaluate

$$(1)\ \int \frac{1}{2+\sin x}dx \quad (2)\ \int e^x \sin x\,dx \quad (3)\ \int \frac{1}{\sqrt{4x-x^2}}\,dx \quad (4)\ \int_0^{\pi/4} \tan^3 x \sec^3 x\,dx$$

$$(5)\ \int_1^2 \frac{1}{e^x - e^{-x}}dx \quad (6)\ \int_0^\infty x^2 e^{-x} dx \quad (7)\ \int_0^1 \frac{1}{\sqrt{1+e^x}}\,dx \quad (8)\ \int_0^\pi \sqrt{1+\sin x}\,dx$$

   Ans:

   (1) With the substitutions $z = \tan\frac{x}{2}$, $z + \frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$,

$$\begin{aligned}\int \frac{1}{2+\sin x}\,dx &= \int \frac{1}{2+\frac{2z}{1+z^2}}\frac{2}{1+z^2}\,dz = \int \frac{1}{z^2+z+1}\,dz = \int \frac{1}{\left(z+\frac{1}{2}\right)^2+\frac{3}{4}}\,dz\\ &= \int \frac{\frac{\sqrt{3}}{2}\sec^2\theta}{\frac{3}{4}(\tan^2\theta+1)}\,d\theta = \int \frac{2}{3}\sqrt{3}\,d\theta = \frac{2}{3}\sqrt{3}\,\theta + C\\ &= \frac{2}{3}\sqrt{3}\,\tan^{-1}\left(\frac{2\tan\frac{x}{2}+1}{\sqrt{3}}\right) + C.\end{aligned}$$

(2) With integration by parts,

$$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx = e^x \sin x - \left(e^x \cos x + \int e^x \sin x\,dx\right).$$

Hence

$$\int e^x \sin x\,dx = \frac{1}{2}e^x \sin x - \frac{1}{2}e^x \cos x + C.$$

(3) With the substitution $x-2 = 2\sin\theta$

$$\int \frac{1}{\sqrt{4x-x^2}}\,dx = \int \frac{1}{\sqrt{-(x-2)^2+4}}\,dx = \int \frac{2\cos\theta}{\sqrt{4-4\sin^2\theta}}\,d\theta$$
$$= \int d\theta = \theta + C = \sin^{-1}\left(\frac{x-2}{2}\right) + C.$$

(4) With the substitution $u = \sec x$,

$$\int_0^{\frac{\pi}{4}} \tan^3 x \sec^3 x\,dx = \int_0^{\frac{\pi}{4}} \left(\sec^2 x - 1\right)\sec^2 x \tan x \sec x\,dx = \int_1^{\sqrt{2}} \left(u^2-1\right)u^2\,du$$
$$= \frac{1}{5}u^5 - \frac{1}{3}u^3\bigg|_1^{\sqrt{2}} = \frac{1}{5}2^{\frac{5}{2}} - \frac{1}{3}2^{\frac{3}{2}} - \frac{2}{15} = \frac{2}{15}\left(\sqrt{2}+1\right).$$

(5) With the substitution $u = e^x$,

$$\int_1^2 \frac{1}{e^x - e^{-x}}\,dx = \int_1^2 \frac{e^x}{e^{2x}-1}\,dx = \int_e^{e^2} \frac{1}{1-u^2}\,du = \int_e^{e^2}\left(\frac{1}{2}\frac{1}{u-1} - \frac{1}{2}\frac{1}{u+1}\right)du$$
$$= \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1|\bigg|_e^{e^2} = \frac{1}{2}\ln\left|\frac{e^2-1}{e^2+1}\right| - \frac{1}{2}\ln\left|\frac{e-1}{e+1}\right|.$$

(6) With integration by parts,

$$\int_0^\infty x^2 e^{-x}\,dx = \lim_{A\to\infty}\int_0^A x^2 e^{-x}\,dx = \lim_{A\to\infty}\left(-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\bigg|_0^A\right) = 2.$$

(7) With the substitution $u = \sqrt{1+e^x}$,

$$\int_0^1 \frac{1}{\sqrt{1+e^x}} = \int_{\sqrt{2}}^{\sqrt{1+e}} \frac{1}{u}\frac{2u}{u^2-1}\,du = \int_{\sqrt{2}}^{\sqrt{1+e}}\left(\frac{1}{u-1} - \frac{1}{u+1}\right)du$$
$$= \ln|u-1| - \ln|u+1|\bigg|_{\sqrt{2}}^{\sqrt{1+e}} = \ln\left|\frac{\sqrt{e+1}-1}{\sqrt{e+1}+1}\right| - \ln\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|.$$

(8) With the substitution $u = \sin x$,

$$\int_0^\pi \sqrt{1+\sin x}\, dx = \int_0^\pi \frac{\sqrt{1+\sin x}\sqrt{1-\sin x}}{\sqrt{1-\sin x}}\, dx = \int_0^\pi \frac{|\cos x|}{\sqrt{1-\sin x}}\, dx$$

$$= \int_0^1 \frac{1}{\sqrt{1-u}}\, du - \int_1^0 \frac{1}{\sqrt{1-u}}\, du = 2\left(-2\sqrt{1-u}\,\Big|_0^1\right) = 4.$$

5. (12 pts) Does the improper integral $\int_1^\infty \frac{1}{\sqrt{x^3-x}} dx$ converge or diverge? Explain (DO NOT try to integrate explicitly).

Ans: Note that $\frac{1}{\sqrt{x^3-x}}$ has two 'improper' parts: $x \to \infty$ and $x = 1^+$.

For the $x = 1^+$ part, let $v = x - 1$,

$$\int_1^2 \frac{1}{\sqrt{x^3-x}}\, dx = \int_0^1 \frac{1}{\sqrt{v(v+1)(v+2)}}\, dv$$

Since

$$\lim_{v \to 0^+} \frac{\frac{1}{\sqrt{v(v+1)(v+2)}}}{\frac{1}{v^{1/2}}} = \frac{1}{\sqrt{2}}$$

and $\int_0^1 \frac{1}{v^{1/2}}\, dv$ converges, by the limit comparison test,

$$\int_1^2 \frac{1}{\sqrt{x^3-x}}\, dx = \int_0^1 \frac{1}{\sqrt{v(v+1)(v+2)}}\, dv$$

converges.

Similarly, since

$$\lim_{x \to \infty} \frac{\frac{1}{\sqrt{x^3-x}}}{\frac{1}{x^{3/2}}} = 1$$

and $\int_2^\infty \frac{1}{x^{3/2}}\, dx$ converges, by the limit comparison test, $\int_2^\infty \frac{1}{\sqrt{x^3-x}}\, dx$ converges. Therefore,

$$\int_1^\infty \frac{1}{\sqrt{x^3-x}}\, dx = \int_1^2 \frac{1}{\sqrt{x^3-x}}\, dx + \int_2^\infty \frac{1}{\sqrt{x^3-x}}\, dx$$

converges.

6. (8 pts) Express $\int_1^2 \sin x\; dx$ as a Riemann sum. That is, $\lim\limits_{\cdots} \sum\limits_{\cdots}^{\cdots} \cdots$.

Ans: Let $P = \{1 = x_0 < x_1 < \cdots < x_n = 2\}$ be a partition of $[1, 2]$ and $\Delta x_k = x_k - x_k - 1$. Then

$$\int_1^2 \sin x\, dx = \lim_{\|P\| \to 0} \sum_{k=1}^n \sin c_k\, \Delta x_k, \quad \text{where } c_k \in [x_{k-1}, x_k].$$

Or consider the partition $P = \left\{1, 1+\frac{1}{n}, \ldots, 1+\frac{n-1}{n}, 2\right\}$. Then

$$\int_1^2 \sin x \, dx = \lim_{n\to\infty} \sum_{k=1}^{n} \sin\left(1+\frac{k}{n}\right)\frac{1}{n} = \lim_{n\to\infty} \sum_{k=n+1}^{2n} \sin\left(\frac{k}{n}\right)\frac{1}{n}.$$

7. (8 pts) Evaluate $\lim_{x\to 0^+} x^{x\ln x}$.

   Ans: Since

$$\lim_{x\to 0^+} x\,(\ln x)^2 = \lim_{x\to 0^+} \frac{(\ln x)^2}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{2\,\ln x\,\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{-2\,\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{-2\,\frac{1}{x}}{-\frac{1}{x^2}} = 0,$$

   we have

$$\lim_{x\to 0^+} x^{x\,\ln x} = \lim_{x\to 0^+} e^{x(\ln x)^2} = e^0 = 1.$$

8. (16 pts) Find the volume and surface area of the object obtained by rotating the region $\{(x-2)^2 + y^2 \leq 1,\ x \geq 2\}$ around the y axis. Note the surface area consists of two parts, one generated by a half circle, the other generated by a line segment.

   Ans: It is the same as Problem 7 of the second midterm exam.

$$V = \frac{4}{3}\pi + 2\pi^2, \qquad A = 4\pi^2 + 12\pi.$$

9. (.100 pts) Schedule recitation change for next semester.

   Ans: The schedule for recitation is changed to Wed evening next semester.