Quiz 5

1. Let $\begin{cases} u = 2x - y \\ v = y \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2}(u+v) \\ y = v \end{cases} \Rightarrow J(u,v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = \frac{1}{2}$

$$\int_0^1 \int_{\frac{y}{2}}^{\frac{y}{2}+1} y^3 (2x-y) e^{(2x-y)^2} \, dx\,dy = \int_0^1 \int_0^2 v^3 u e^{u^2} \cdot \frac{1}{2} \, du\,dv$$

$$= \int_0^1 \frac{1}{4} v^3 e^{u^2} \Big|_{u=0}^{u=2} dv = \int_0^1 \frac{1}{4} v^3 (e^4 - 1)\, dv = \frac{1}{16}(e^4-1) v^4 \Big|_0^1 = \frac{1}{16}(e^4-1)$$

2. (i) $\vec{r}(t) = (1-t,\ 1-t,\ 1-t)$, $0 \le t \le 1 \Rightarrow \frac{d\vec{r}(t)}{dt} = (-1,-1,-1)$

$$\int_C \vec{F}\, d\vec{r} = \int_0^1 ((1-t)(1-t),\ (1-t)(1-t),\ (1-t)^2) \cdot (-1,-1,-1)\, dt$$

$$= \int_0^1 -3(1-t)^2\, dt = (1-t)^3 \Big|_0^1 = -1$$

(ii) $\because \frac{\partial(xy)}{\partial y} = x$ but $\frac{\partial(yz)}{\partial x} = 0 \Rightarrow \frac{\partial(xy)}{\partial y} \ne \frac{\partial(yz)}{\partial x}$ when $x \ne 0$

$\Rightarrow \vec{F}$ is not conservative

$\Rightarrow$ The answer may be different if we change it to a different path

3. $\because \frac{\partial(z+y)}{\partial y} = 1$ but $\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial(z+y)}{\partial y} \ne \frac{\partial z}{\partial x}$

$\Rightarrow \vec{F}$ is not conservative

4. Let $C: \vec{r}(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi \Rightarrow \frac{d\vec{r}(t)}{dt} = (-\sin t, \cos t)$

$\Omega = \{(x,y) \mid x^2 + y^2 < 1\}$

(i) Green's Theorem (Normal Form)

$$\oint_C (xy, y^2) \cdot \vec{n}\, ds = \iint_\Omega \left(\frac{\partial(xy)}{\partial x} + \frac{\partial(y^2)}{\partial y} \right) dx\,dy$$

Where $\vec{n}$ is outward unit normal

check:

$$\oint_C (xy, y^2)\cdot\vec{n}\,ds = \int_0^{2\pi} ((\cos t)(\sin t), (\sin t)^2)\cdot(\cos t, \sin t)\,dt$$

$$= \int_0^{2\pi} \sin t\cos^2 t + \sin^3 t\,dt = \int_0^{2\pi} \sin t\,dt = 0$$

and

$$\iint_\Omega \left(\frac{\partial(xy)}{\partial x} + \frac{\partial(y^2)}{\partial y}\right)dxdy = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3y\,dy\,dx = \int_{-1}^{1} \frac{3}{2}y^2\Big|_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}}\,dx = \int_{-1}^{1} 0\,dx = 0$$

$\Rightarrow$ They are the same

(ii) Green's Theorem (Tangential Form)

$$\oint_C (xy, y^2)\cdot\vec{T}\,ds = \iint_\Omega \left(\frac{\partial(y^2)}{\partial x} - \frac{\partial(xy)}{\partial y}\right)dxdy$$

where $\vec{T}$ is unit tangent

check:

$$\oint_C (xy, y^2)\cdot\vec{T}\,ds = \int_0^{2\pi} (\sin t\cos t, \sin^2 t)\cdot(-\sin t, \cos t)\,dt$$

$$= \int_0^{2\pi} 0\,dt = 0$$

and

$$\iint_\Omega \left(\frac{\partial(y^2)}{\partial x} - \frac{\partial(xy)}{\partial y}\right)dxdy = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} -x\,dxdy = \int_{-1}^{1} -\frac{x^2}{2}\Big|_{x=-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}}\,dy = \int_{-1}^{1} 0\,dy = 0$$

$\Rightarrow$ They are the same