Quiz 4

1. tangent plan : $f_x(0,0)(x-0)+f_y(0,0)(y-0)+(-1)(z-f(0,0))=0$

$$\Rightarrow z = L(x,y) = f(0,0) + x\, f_x(0,0) + y\, f_y(0,0)$$

2. Let $g(t) = f(xt, yt)$

$$\because \quad g(t) = g(0) + g'(0)(t-0) + \frac{g''(0)}{2!}(t-0)^2 + \frac{g'''(c)}{3!}(t-0)^3$$

for some c between 0 and 1

Since $g'(0) = \left.\frac{d f(xt,yt)}{dt}\right|_{t=0} = \left.\left(f_x(xt,yt)\cdot\frac{d(xt)}{dt} + f_y(xt,yt)\frac{d(yt)}{dt}\right)\right|_{t=0}$

$$= f_x(0,0)\cdot x + f_y(0,0)\cdot y$$

and $g''(0) = \left.\frac{d}{dt}\left(f_x(xt,yt)\cdot x + f_y(xt,yt)\cdot y\right)\right|_{t=0}$

$$= \Big(x\cdot\big(f_{xx}(xt,yt)\cdot\tfrac{d(xt)}{dt} + f_{xy}(xt,yt)\cdot\tfrac{d(yt)}{dt}\big) + y\cdot\big(f_{yx}(xt,yt)\cdot\tfrac{d(xt)}{dt} + f_{yy}(xt,yt)\cdot\tfrac{d(yt)}{dt}\big)\Big)\Big|_{t=0}$$

$$= f_{xx}(0,0)\cdot x^2 + 2f_{xy}(0,0)\cdot xy + f_{yy}(0,0)\cdot y^2$$

Same process for $g'''(c)$, we can derive

$f(x,y) = g(1) = g(0) + g'(0) + \frac{g''(0)}{2!} + \frac{g'''(c)}{3!}$ for some c between 0 & 1

$$= P_2(x,y) + R_2(x,y)$$

$$P_2(x,y) = g(0) + g'(0) + \frac{1}{2!}g''(0)$$

$$= f(0,0) + f_x(0,0)\,x + f_y(0,0)\,y + \frac{1}{2!}\left(f_{xx}(0,0)\,x^2 + 2f_{xy}(0,0)\,xy + f_{yy}(0,0)\,y^2\right)$$

$$R_2(x,y)=\frac{1}{3!}\left(f_{xxx}(xc,yc)x^3+3f_{xxy}(xc,yc)x^2y+3f_{xyy}(xc,yc)xy^2+f_{yyy}(xc,yc)y^3\right)$$

for some c between 0 and 1

3. $f(x,y,z)=xy+z^2$ on $x^2+y^2+z^2=16$

Let $g(x,y,z)=x^2+y^2+z^2-16$

$$\begin{cases}\nabla f=\lambda\nabla g\\ g=0\end{cases}\Rightarrow\begin{cases}y=2\lambda x\\ x=2\lambda y\\ 2z=2\lambda z\\ x^2+y^2+z^2=16\end{cases}$$

(i) if $\lambda=0\Rightarrow x=y=z=0\Rightarrow x^2+y^2+z^2\neq16\Rightarrow\lambda\neq0$

(ii) $\lambda\neq0$, $\begin{cases}y=2\lambda x\\ x=2\lambda y\end{cases}\Rightarrow 2\lambda y^2=2\lambda x^2\Rightarrow y^2=x^2$

① $y=x\Rightarrow x(1-2\lambda)=0\Rightarrow x=0$ or $\lambda=\frac{1}{2}$

(a) $x=0\Rightarrow y=0$, $z=\pm4$

(b) $\lambda=\frac{1}{2}\Rightarrow z=0\Rightarrow x=y=\pm2\sqrt{2}$

② $y=-x\Rightarrow x(1+2\lambda)=0\Rightarrow x=0$ or $\lambda=-\frac{1}{2}$

(a) $x=0\Rightarrow y=0$, $z=\pm4$

(b) $\lambda=-\frac{1}{2}\Rightarrow z=0\Rightarrow x=\pm2\sqrt{2}$, $y=-x$

$\Rightarrow(0,0,\pm4)$, $(2\sqrt{2},2\sqrt{2},0)$, $(-2\sqrt{2},-2\sqrt{2},0)$, $(2\sqrt{2},-2\sqrt{2},0)$, $(-2\sqrt{2},2\sqrt{2},0)$

$\because$ $f(0,0,\pm 4)=16$, $f(2\sqrt{2},2\sqrt{2},0)=f(-2\sqrt{2},-2\sqrt{2},0)=8$

$f(2\sqrt{2},-2\sqrt{2},0)=f(-2\sqrt{2},2\sqrt{2},0)=-8$

$\therefore$ f has absolute maximum 16 at $(0,0,\pm 4)$

absolute minimum -8 at $(2\sqrt{2},-2\sqrt{2},0)$, $(-2\sqrt{2},2\sqrt{2},0)$

4. Let $f(x,y,z)=x^2+2y^2+3z^2$, $g(x,y,z)=x+y-z$

$\Rightarrow$ $\nabla f(1,1,-1)=(2,4,-6)$, $\nabla g(1,1,-1)=(1,1,-1)$

$\Rightarrow$ $\nabla f(1,1,-1)\times\nabla g(1,1,-1)=(2,-4,-2)$

$\Rightarrow$ equation of plane : $2(x-1)-4(y-1)-2(z+1)=0$