Quiz 2

1. $\sum_{n=2}^{\infty} \frac{x^n}{n\ln n} \Rightarrow a_n = \frac{x^n}{n\ln n}$

(i) $\because \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{\ln n}{\ln(n+1)} \cdot |x| = |x|$

$\therefore$ By Ratio test, the series converges absolutely on $|x|<1$

(ii) when $x=1$, $\sum_{n=2}^{\infty} \frac{x^n}{n\ln n} = \sum_{n=2}^{\infty} \frac{1}{n\ln n}$

$\because$ $f(x) = \frac{1}{x\ln x}$ is continuous, positive, decreasing for all $x \geq 2$

and $\int_2^{\infty} \frac{1}{x\ln x}\,dx$ diverges

$\therefore$ By Integral test, the series diverges when $x=1$

(iii) when $x=-1$, $\sum_{n=2}^{\infty} \frac{x^n}{n\ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln n} \Rightarrow b_n = \frac{1}{n\ln n}$

$\because \lim_{n\to\infty} b_n = 0$, $b_n > 0$ and $b_n > b_{n+1}$, $\forall n \geq 2$

$\therefore$ By Alternating Series theorem, the series converges when $x=-1$

By (i), (ii) and (iii), the series converges when $-1 \leq x < 1$

2. Taylor's Theorem for f: $\forall a \in \mathbb{R}$, for each positive integer n

and for each $x \in \mathbb{R}$,

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x

Derive $R_2(X)$:

$\forall\, b \in \mathbb{R}$, let $P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$

and $\phi_2(x) = P_2(x) + \frac{f(b) - P_2(b)}{(b-a)^3}(x-a)^3$

Hence $\phi_2(a) = P_2(a) = f(a)$ and $\phi_2(b) = f(b)$

Moreover, $\phi_2(x)$ is infinitely many times differentiable on $\mathbb{R}$

Let $F(x) = f(x) - \phi_2(x)$ on $\mathbb{R}$

$\Rightarrow$ $F(x)$ is infinitely many times differentiable on $\mathbb{R}$

$\because$ $F(a) = F(b) = 0$

By Rolle's theorem, there exists c_1 between a and b s.t.

$F'(c_1) = 0$

$\because$ $F'(a) = F'(c_1) = 0$

By Rolle's theorem, there exists c_2 between a and c_1 s.t.

$F''(c_2) = 0$

$\because$ $F''(a) = F''(c_2) = 0$

By Rolle's theorem, there exists c between a and c_2 s.t.

$F'''(c) = 0$

$$\Rightarrow \quad 0 = F'''(c) = f'''(c) - \frac{f(b)-P_2(b)}{(b-a)^3}\cdot 3!$$

$$\Rightarrow \quad f(b) = P_2(b) + \frac{f'''(c)}{3!}(b-a)^3$$

$$\Rightarrow \quad \forall x \in \mathbb{R}, \quad f(x) = P_2(x) + \frac{f'''(c)}{3!}(x-a)^3$$

$$\Rightarrow \quad R_2(x) = \frac{f'''(c)}{3!}(x-a)^3$$, for some c between a and x

3. Taylor series generated by $\sin x$ at $x=0$ is

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Since $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$, for some c between 0 and x

and $|f^{(n+1)}(c)| \le 1$, $\forall c \in \mathbb{R}$

$$\Rightarrow \quad 0 \le |R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!}$$

$\because \lim\limits_{n\to\infty} \dfrac{|x|^{n+1}}{(n+1)!} = 0$, $\forall x \in \mathbb{R}$, $\therefore \lim\limits_{n\to\infty} R_n(x) = 0$, $\forall x \in \mathbb{R}$

Hence $\sin x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$, $\forall x \in \mathbb{R}$

4. Let $f(x) = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}x^n}{n\,2^n}$, $-2 < x < 2$

$$\Rightarrow \quad f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{n-1}}{2^n} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{-x}{2}\right)^{n-1} \quad , -2<x<2$$

$$= \frac{1}{2}\cdot\frac{1}{1-(-\frac{x}{2})} = \frac{1}{2+x} \quad , -2<x<2$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n\cdot 2^n} = f(x) = f(0) + \int_0^x f'(t)\,dt = 0 + \ln|2+t|\Big|_0^x$$

$$= \ln\left|\frac{2+x}{2}\right| \quad , -2<x<2$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot 2^n} = f(1) = \ln \frac{3}{2}$$

5.

$$f(x) = \begin{cases} 0 & , x = 0 \\ e^{-\frac{1}{x^4}} & , x \neq 0 \end{cases}$$

when $x \neq 0$, $f'(x) = \frac{4}{x^5} e^{-\frac{1}{x^4}}$.

$\Rightarrow f(x)$ is differentiable as $x \neq 0$

Moreover, $f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{e^{-\frac{1}{h^4}}}{h}$

$$= \lim_{h \to 0} \frac{\frac{1}{h}}{e^{\frac{1}{h^4}}} = \lim_{h \to 0} \frac{\frac{-1}{h^2}}{e^{\frac{1}{h^4}} \cdot \frac{(-4)}{h^5}} = \lim_{h \to 0} \frac{h^3}{4 e^{\frac{1}{h^4}}} = 0$$

$\Rightarrow f(x)$ is differentiable as $x = 0$

$\Rightarrow f$ is differentiable everywhere.