Midterm 2

1. (a) $\left(\frac{df}{ds}\right)_{u^\theta,P} = \lim_{h\to 0} \frac{f((0,0)+hu^\theta)-f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{h^3\cos^2\theta\sin\theta}{h^2}-0}{h} = \cos^2\theta\sin\theta$

(b) $f_x(0,0) = \left(\frac{df}{ds}\right)_{u^\theta,P}\Big|_{\theta=0} = 0$, $f_y(0,0) = \left(\frac{df}{ds}\right)_{u^\theta,P}\Big|_{\theta=\frac{\pi}{2}} = 0$

$\Rightarrow L(x,y) = f(0,0) + x f_x(0,0) + y f_y(0,0) = 0$

$\because \lim_{\substack{y=x\\ x\to 0^+}} \frac{f(x,y)-L(x,y)}{\sqrt{x^2+y^2}} = \lim_{x\to 0}\frac{\frac{x^3}{2x^2}}{\sqrt{2}x} = \frac{1}{2\sqrt{2}}$

$\therefore \lim_{(x,y)\to(0,0)} \frac{f(x,y)-L(x,y)}{\sqrt{x^2+y^2}} \neq 0 \Rightarrow f$ is not differentiable at $(0,0)$

$\Rightarrow f$ doesn't have a linear approximation near $(0,0)$

2. Let $g_1(x,y,z) = x^2+y^2+z^2-9$, $g_2(x,y,z) = x-y$

$$\begin{cases} \nabla f = \lambda\nabla g_1 + \mu\nabla g_2 \\ g_1 = 0 \\ g_2 = 0 \end{cases} \Rightarrow \begin{cases} \begin{pmatrix} y \\ x \\ 4z \end{pmatrix} = \lambda\begin{pmatrix} 2x \\ 2y \\ 2z\end{pmatrix} + \mu\begin{pmatrix} 1 \\ -1 \\ 0\end{pmatrix} \\ x^2+y^2+z^2-9=0 \\ x-y=0 \end{cases}$$

$$\begin{cases} y = 2\lambda x + \mu \\ x = 2\lambda y - \mu \end{cases} \Rightarrow x+y = 2\lambda(x+y) \Rightarrow (x+y)(2\lambda-1)=0$$

$\Rightarrow y=-x$ or $\lambda = \frac{1}{2}$

① $y=-x$, $\because x-y=0 \Rightarrow x=y=0 \Rightarrow z=\pm 3 \Rightarrow (0,0,\pm 3)$

② $\lambda=\frac{1}{2} \Rightarrow 4z = 2\lambda z = z \Rightarrow z=0$, $x=y \Rightarrow x^2+y^2+z^2-9 = 2x^2-9=0$

$\Rightarrow x = \pm\frac{3}{\sqrt{2}} \Rightarrow (\frac{3}{\sqrt{2}},\frac{3}{\sqrt{2}},0)$, $(-\frac{3}{\sqrt{2}},-\frac{3}{\sqrt{2}},0)$

Since $f(\frac{3}{\sqrt{2}},\frac{3}{\sqrt{2}},0) = f(-\frac{3}{\sqrt{2}},-\frac{3}{\sqrt{2}},0) = \frac{9}{2}$, $f(0,0,\pm3) = 18$

f has absolute maximum 18 at $(0,0,\pm 3)$

absolute minimum $\frac{9}{2}$ at $(\frac{3}{\sqrt{2}},\frac{3}{\sqrt{2}},0)$, $(-\frac{3}{\sqrt{2}},-\frac{3}{\sqrt{2}},0)$

3. (a) Suppose $f(x,y)$ and all its first and second derivatives are continuous in $\mathbb{R}^2$ and $f_x(a,b)=f_y(a,b)=0$, then

① If $f_{xx}f_{yy}-f_{xy}^2>0$ and $f_{xx}<0$ at (a,b), then f has a local maximum at (a,b)

② If $f_{xx}f_{yy}-f_{xy}^2>0$ and $f_{xx}>0$ at (a,b), then f has a local minimum at (a,b)

③ If $f_{xx}f_{yy}-f_{xy}^2<0$ at (a,b), then f has a saddle point at (a,b)

④ If $f_{xx}f_{yy}-f_{xy}^2=0$ at (a,b), the test is inconclusive at (a,b)

(b) Let $g(x,y,z)=x^2-y^2$

then $g_x(0,0,0)=g_y(0,0,0)=g_z(0,0,0)=0$

and g has local minimum at $(0,0,0)$ on x-axis

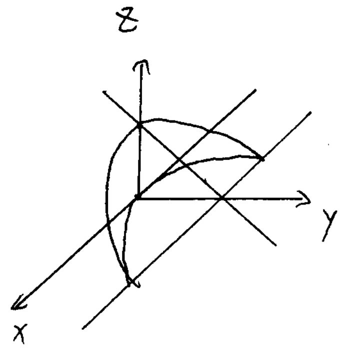
local maximum at $(0,0,0)$ on y-axis

$\Rightarrow g$ has a saddle point at $(0,0,0)$

4. $\int_0^2\int_y^2 x^2\cos(xy)\,dx\,dy = \int_0^2\int_0^x x^2\cos(xy)\,dy\,dx = \int_0^2 x\sin(xy)\Big|_{y=0}^{y=x}\,dx$

$= \int_0^2 x\sin(x^2)\,dx = -\frac{1}{2}\cos(x^2)\Big|_0^2 = \frac{1}{2}(1-\cos 4)$

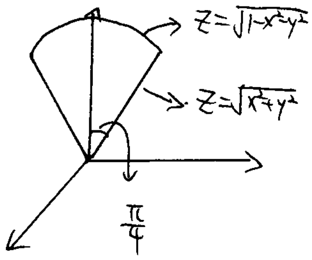
5. $\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} dz\,dy\,dx \Rightarrow$



$$= \int_{0}^{1}\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{x^2}^{1-z} dy\,dx\,dz$$

$$= \int_{0}^{1}\int_{0}^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy\,dz$$

6. $\{\rho \le 1\} \cap \{\phi \le \frac{\pi}{4}\} \Rightarrow$



$$\text{volume} = \int_{0}^{2\pi}\int_{0}^{1}\int_{r}^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta$$

$$= \int_{0}^{2\pi}\int_{0}^{\frac{\pi}{4}}\int_{0}^{1} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_{0}^{2\pi}\int_{0}^{\frac{\pi}{4}} \frac{1}{3}\sin\phi\,d\phi\,d\theta = \int_{0}^{2\pi}\frac{1}{3}\left(1-\frac{1}{\sqrt{2}}\right)d\theta = \frac{2\pi}{3}\left(1-\frac{1}{\sqrt{2}}\right)$$

7. $$P_2(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$$

$$+\frac{1}{2!}\Big(f_{xx}(x_0,y_0,z_0)(x-x_0)^2 + f_{yy}(x_0,y_0,z_0)(y-y_0)^2 + f_{zz}(x_0,y_0,z_0)(z-z_0)^2$$
$$+2f_{xy}(x_0,y_0,z_0)(x-x_0)(y-y_0) + 2f_{xz}(x_0,y_0,z_0)(x-x_0)(z-z_0) + 2f_{yz}(x_0,y_0,z_0)(y-y_0)(z-z_0)\Big)$$

$$R_2(x,y,z) = \frac{1}{3!}\Big(f_{xxx}(x_c,y_c,z_c)(x-x_0)^3 + f_{yyy}(x_c,y_c,z_c)(y-y_0)^3 + f_{zzz}(x_c,y_c,z_c)(z-z_0)^3$$

$$+3f_{xxy}(x_c,y_c,z_c)(x-x_0)^2(y-y_0) + 3f_{xxz}(x_c,y_c,z_c)(x-x_0)^2(z-z_0)$$

$$+3f_{xyy}(x_c,y_c,z_c)(x-x_0)(y-y_0)^2 + 3f_{yyz}(x_c,y_c,z_c)(y-y_0)^2(z-z_0)$$

$$+3f_{xzz}(x_c,y_c,z_c)(x-x_0)(z-z_0)^2 + 3f_{yzz}(x_c,y_c,z_c)(y-y_0)(z-z_0)^2$$

$$+6f_{xyz}(x_c,y_c,z_c)(x-x_0)(y-y_0)(z-z_0)\Big)$$

for some (x_c, y_c, z_c) lies on the line segment between (x_0, y_0, z_0) and (x, y, z)