Midterm 1

1. $\because \lim_{x\to\infty} \dfrac{\frac{1}{\sqrt[3]{x+x^2}}}{\frac{1}{x^{\frac{2}{3}}}} = \lim_{x\to\infty} \dfrac{1}{\sqrt[3]{\frac{1}{x}+1}} = 1$, and $\int_1^\infty \frac{1}{x^{\frac{2}{3}}}\,dx$ diverges

By Limit Comparison test , $\int_1^\infty \frac{1}{\sqrt[3]{x+x^2}}\,dx$ diverges

$\Rightarrow \int_0^\infty \frac{1}{\sqrt[3]{x+x^2}}\,dx$ diverges

2. (a) $\because \sum_{n=0}^{\infty} a_n(-2)^n$ converges $\Rightarrow \lim_{n\to\infty} a_n(-2)^n = 0$

$\Rightarrow \exists\, k \in \mathbb{N}$ s.t. $|a_n(-2)^n| < 1$, $\forall n \geq k$

$\Rightarrow |a_n| < \frac{1}{2^n}$, $\forall n \geq k$

$\Rightarrow \sum_{n=k}^{\infty} |a_n| < \sum_{n=k}^{\infty} \frac{1}{2^n} = \frac{1}{2^{k-1}}$

$\Rightarrow \sum_{n=k}^{\infty} |a_n|$ converges $\Rightarrow \sum_{n=0}^{\infty} |a_n|$ converges

$\Rightarrow \sum_{n=0}^{\infty} a_n$ converges absolutely $\Rightarrow \sum_{n=0}^{\infty} a_n x^n$ converges absolutely at $x=1$

$\Rightarrow$ True

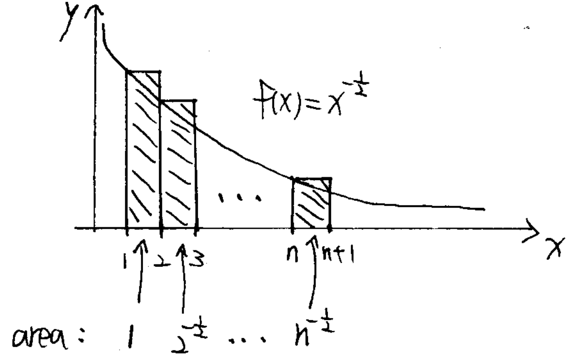
(b) Let $a_n = \frac{1}{n+1}$, $n = 0, 1, 2, \cdots$

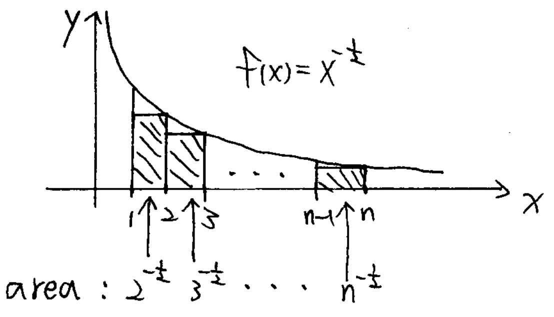
(i) when $x=-1$, $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ converges by Leibniz's Theorem

(ii) when $x=1$, $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges (harmonic series)

$\Rightarrow$ False

3. $\because \int_1^{n+1} x^{-\frac{1}{2}}\,dx < \sum_{k=1}^{n} k^{-\frac{1}{2}} < 1 + \int_1^{n} x^{-\frac{1}{2}}\,dx$





$\Rightarrow 2(\sqrt{n+1}-1) < \sum_{k=1}^{n} k^{-\frac{1}{2}} < 2\sqrt{n} - 1$

$\Rightarrow \dfrac{\ln[2(\sqrt{n+1}-1)]}{\ln n} < \dfrac{\ln\left(\sum_{k=1}^{n} k^{-\frac{1}{2}}\right)}{\ln n} < \dfrac{\ln(2\sqrt{n}-1)}{\ln n}$

$\because \lim_{n\to\infty} \dfrac{\ln[2(\sqrt{n+1}-1)]}{\ln n} = \lim_{n\to\infty} \dfrac{\frac{1}{2(\sqrt{n+1}-1)}\cdot\frac{1}{\sqrt{n+1}}}{\frac{1}{n}}$

$= \lim_{n\to\infty} \dfrac{n}{2\sqrt{n+1}(\sqrt{n+1}-1)} = \lim_{n\to\infty} \dfrac{1}{2\sqrt{1+\frac{1}{n}}\left(\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}\right)} = \dfrac{1}{2}$

and

$\lim_{n\to\infty} \dfrac{\ln(2\sqrt{n}-1)}{\ln n} = \lim_{n\to\infty} \dfrac{\frac{1}{2\sqrt{n}-1}\cdot\frac{1}{\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \dfrac{n}{\sqrt{n}(2\sqrt{n}-1)} = \lim_{n\to\infty} \dfrac{1}{\left(2-\frac{1}{\sqrt{n}}\right)} = \dfrac{1}{2}$

By Sandwich Theorem, $\lim_{n\to\infty} \dfrac{\ln\left(\sum_{k=1}^{n} k^{-\frac{1}{2}}\right)}{\ln n} = \dfrac{1}{2}$

4. (a) $\because \lim_{n\to\infty} \dfrac{\frac{(\frac{\pi}{2})^{2n+2}}{(2n+2)!}}{\frac{(\frac{\pi}{2})^{2n}}{(2n)!}} = \lim_{n\to\infty} \dfrac{(\frac{\pi}{2})^2}{(2n+1)(2n+2)} = 0 < 1$

By Ratio test, $\sum_{n=0}^{\infty} \dfrac{(\frac{\pi}{2})^{2n}}{(2n)!}$ converges $\Rightarrow \sum_{n=0}^{\infty} (-1)^n \dfrac{\pi^{2n}}{2^{2n}(2n)!}$ converges absolutely

(b) $\because \cos x = \sum_{k=0}^{n} (-1)^k \dfrac{x^{2k}}{(2k)!} + \dfrac{\cos^{(2k+1)}(c)}{(2k+1)!} x^{2k+1}$, for some c between 0 and x.

Since $0 \le \left| \dfrac{\cos^{(2k+1)}(c)}{(2k+1)!} x^{2k+1} \right| \le \dfrac{|x|^{2k+1}}{(2k+1)!}$, and $\lim_{k\to\infty} \dfrac{|x|^{2k+1}}{(2k+1)!} = 0$, $\forall x \in \mathbb{R}$

By Sandwich Theorem, $\lim_{n\to\infty} \dfrac{\cos^{(2k+1)}(c)}{(2k+1)!} x^{2k+1} = 0$, $\forall x \in \mathbb{R}$

$\Rightarrow \cos x = \sum_{k=0}^{\infty} (-1)^k \dfrac{x^{2k}}{(2k)!}$

$\Rightarrow \sum_{k=0}^{\infty} (-1)^k \dfrac{\pi^{2k}}{2^{2k}(2k)!} = \cos\left(\frac{\pi}{2}\right) = 0$

5. Assume $y(x)=a_0+a_1x+a_2x^2+\cdots$

$$y(0)=1 \Rightarrow a_0=1 \Rightarrow y-1=a_1x+a_2x^2+\cdots$$

$$\Rightarrow \cos(y-1)=1-\frac{1}{2!}(y-1)^2+\frac{1}{4!}(y-1)^4-\cdots$$

$$=1-\frac{a_1^2}{2}x^2-a_1a_2x^3+\cdots$$

$$\because \frac{dy}{dx}=y+\cos(y-1)$$

$$\Rightarrow a_1+2a_2x+3a_3x^2+\cdots=(1+a_1x+a_2x^2+\cdots)+(1-\frac{a_1^2}{2}x^2-a_1a_2x^3+\cdots)$$

x^0 : $a_1=2$

x^1 : $2a_2=a_1 \Rightarrow a_2=1$

x^2 : $3a_3=a_2-\frac{a_1^2}{2} \Rightarrow a_3=-\frac{1}{3}$

Hence $a_0=1$, $a_1=2$, $a_2=1$, $a_3=-\frac{1}{3}$

6. (a) $\because f(x)=a_0+a_1x+a_2x^2+\cdots$ on $|x|<1$

$$\Rightarrow f^{(n)}(x)=n!a_n+(n+1)!a_{n+1}x+\frac{(n+2)!}{2!}a_{n+2}x^2+\cdots \text{ on } |x|<1, \quad \forall n=0,1,2,\cdots$$

$$\Rightarrow f^{(n)}(0)=n!\,a_n \Rightarrow a_n=\frac{f^{(n)}(0)}{n!} \Rightarrow \text{True}$$

(b) Let $f(x)=\begin{cases}0, & x=0\\ e^{-\frac{1}{x^2}}, & x\neq 0\end{cases} \Rightarrow f^{(n)}(0)=0,\ \forall n\in\mathbb{N}$

$$\Rightarrow g(x)=f(0)+\sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=0 \text{ on } |x|<1$$

But $g(x) \neq f(x)$ when $x \neq 0$ $\Rightarrow$ False

7. Since $\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$, $x \in \mathbb{R}$

$$\Rightarrow \int_0^{\frac{1}{2}} \sin(x^2)\,dx = \frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \cdots \Big|_0^{\frac{1}{2}}$$

$$= \frac{(\frac{1}{2})^3}{3} - \frac{(\frac{1}{2})^7}{7\cdot 3!} + \frac{(\frac{1}{2})^{11}}{11\cdot 5!} - \cdots$$

$$\because \frac{(\frac{1}{2})^{15}}{15\cdot 7!} = \frac{1}{2^{15}\cdot 15\cdot 7!} = \frac{1}{1024\cdot 32\cdot 15\cdot 42\cdot 120} < \frac{1}{10^3\cdot 10\cdot 10\cdot 10\cdot 10^2} = 10^{-8}$$

By Alternating Series Estimation Theorem

$$\left| \int_0^{\frac{1}{2}} \sin(x^2)\,dx - \left(\frac{(\frac{1}{2})^3}{3} - \frac{(\frac{1}{2})^7}{7\cdot 3!} + \frac{(\frac{1}{2})^{11}}{11\cdot 5!} \right) \right| < \frac{(\frac{1}{2})^{15}}{15\cdot 7!} < 10^{-8}$$

$$\Rightarrow \text{Approximation}: \frac{(\frac{1}{2})^3}{3} - \frac{(\frac{1}{2})^7}{7\cdot 3!} + \frac{(\frac{1}{2})^{11}}{11\cdot 5!}$$

8 $r = e^{2\theta}$ and $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$\Rightarrow \begin{cases} \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta + r(-\sin\theta) = 2e^{2\theta}\cos\theta - e^{2\theta}\sin\theta \\ \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta = 2e^{2\theta}\sin\theta + e^{2\theta}\cos\theta \end{cases}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2\sin\theta + \cos\theta}{2\cos\theta - \sin\theta} \Rightarrow \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}} = -2$$

$$\Rightarrow \frac{d\left(\frac{dy}{dx}\right)}{d\theta} = \frac{(2\cos\theta - \sin\theta)^2 + (2\sin\theta + \cos\theta)^2}{(2\cos\theta - \sin\theta)^2}$$

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{\theta=\frac{\pi}{2}} = \left.\frac{\frac{d\left(\frac{dy}{dx}\right)}{d\theta}}{\frac{dx}{d\theta}}\right|_{\theta=\frac{\pi}{2}} = \frac{5}{-e^{\pi}}$$