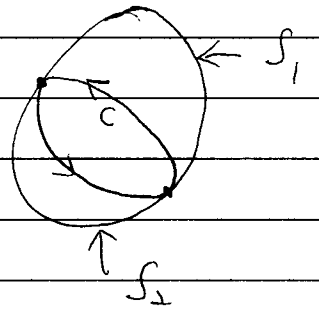
15.7-10 Let $C: \vec{r}(t) = (\cos t, \sin t, 0)$, $0 \le t \le 2\pi \Rightarrow \frac{d\vec{r}}{dt} = (-\sin t, \cos t, 0)$

$$\iint_S \nabla \times (y\hat{i}) \cdot \vec{n}\, d\sigma = \oint_C (y,0,0) \cdot d\vec{r} = \int_0^{2\pi} (\sin t, 0, 0) \cdot \frac{d\vec{r}}{dt}\, dt$$

$$= \int_0^{2\pi} -\sin^2 t\, dt = -\int_0^{2\pi} \frac{1-\cos 2t}{2}\, dt = -\pi$$

16 $\iint_S \nabla \times \vec{F} \cdot \vec{n}\, d\sigma$

$$= \iint_{S_1} \nabla \times \vec{F} \cdot \vec{n}\, d\sigma + \iint_{S_2} \nabla \times \vec{F} \cdot \vec{n}\, d\sigma$$



$$= \oint_C \vec{F} \cdot d\vec{r} + \oint_C \vec{F} \cdot d\vec{r}$$

$$= \oint_C \vec{F} \cdot d\vec{r} - \oint_C \vec{F} \cdot d\vec{r} = 0$$

15.8-18 $\because \iint_{\text{side } y=0} \vec{F} \cdot \vec{n}\, d\sigma + \iint_{\text{side } x=0} \vec{F} \cdot \vec{n}\, d\sigma + \iint_{\text{side } z=0} \vec{F} \cdot \vec{n}\, d\sigma + \iint_{\text{side } A} \vec{F} \cdot \vec{n}\, d\sigma + \iint_{\text{side } B} \vec{F} \cdot \vec{n}\, d\sigma + \iint_{\text{Top}} \vec{F} \cdot \vec{n}\, d\sigma$

$$= \iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iiint_D \nabla \cdot \vec{F}\, dV = \iiint_D 1-2+1\, dV = 0$$

and $\iint_{\text{side } y=0} \vec{F} \cdot \vec{n}\, d\sigma = \iint_{\text{side } y=0} (x, -2y, z+3) \cdot (0,-1,0)\, d\sigma = \iint_{\text{side } y=0} 2y\, d\sigma = \iint_{\text{side } y=0} 0\, d\sigma = 0$

$$\iint_{\text{side } x=0} \vec{F} \cdot \vec{n}\, d\sigma = \iint_{\text{side } x=0} (x, -2y, z+3) \cdot (-1,0,0)\, d\sigma = \iint_{\text{side } x=0} -x\, d\sigma = \iint_{\text{side } x=0} 0\, d\sigma = 0$$

$$\iint_{\text{side } z=0} \vec{F} \cdot \vec{n}\, d\sigma = \iint_{\text{side } z=0} (x, -2y, z+3) \cdot (0,0,-1)\, d\sigma = \iint_{\text{side } z=0} -z-3\, dx\,dy$$

$$= \int_0^1\int_0^1 -3\, dx\,dy = -3$$

$\therefore\ 0+0-3+1-3+\iint_{Top} \vec{F}\cdot\vec{n}\,d\sigma = 0$

$\Rightarrow \iint_{Top} \vec{F}\cdot\vec{n}\,d\sigma = 5$

17. $\left|\iiint_D \nabla\cdot\vec{F}\,dV\right| = \left|\iint_S \vec{F}\cdot\vec{n}\,d\sigma\right| \le \iint_S |\vec{F}\cdot\vec{n}|\,d\sigma \le \iint_S |\vec{F}||\vec{n}|\,d\sigma \le \iint_S 1\,d\sigma$

$= \text{area of } S$

19. $\vec{F} = \vec{C}$: constant vector $\Rightarrow \nabla\cdot\vec{F} = 0$

$\Rightarrow \iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iiint_D \nabla\cdot\vec{F}\,dV = 0$

21. (b) $\because\ \nabla\cdot(\nabla\times\vec{G}) = 0$

$\therefore \iint_S (\nabla\times\vec{G})\cdot\vec{n}\,d\sigma = \iiint_D \nabla\cdot(\nabla\times\vec{G})\,dV = 0$

24. (a) If f is harmonic $\Rightarrow \nabla\cdot(\nabla f) = 0$

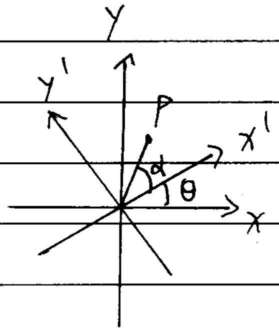
$\Rightarrow \iint_S \nabla f\cdot\vec{n}\,d\sigma = \iiint_D \nabla\cdot(\nabla f)\,dV = 0$

(b) If f is harmonic,

$\iint_S (f\,\nabla f)\cdot\vec{n}\,d\sigma = \iiint_D \nabla\cdot(f\nabla f)\,dV = \iiint_D (\nabla f)\cdot(\nabla f) + f(\nabla\cdot(\nabla f))\,dV$

$= \iiint_D |\nabla f|^2\,dV$

Problem 3 (a) $P = (x_p, y_p)$ in x-y coordinate



$= (x_p', y_p')$ in x'-y' coordinate

$(x_p', y_p') = (\cos\alpha, \sin\alpha)$

$= (\cos(\alpha+\theta-\theta), \sin(\alpha+\theta-\theta))$

$= (\cos(\alpha+\theta)\cos\theta + \sin(\alpha+\theta)\sin\theta, \sin(\alpha+\theta)\cos\theta - \cos(\alpha+\theta)\sin\theta)$

and $(x_p, y_p) = (\cos(\alpha+\theta), \sin(\alpha+\theta))$

hence $x_p' = \cos\theta\, x_p + \sin\theta\, y_p$

$y_p' = -\sin\theta\, x_p + \cos\theta\, y_p$

$$\Rightarrow \begin{cases} x' = \cos\theta\, x + \sin\theta\, y \\ y' = -\sin\theta\, x + \cos\theta\, y \end{cases} \Rightarrow \begin{cases} x = \cos\theta\, x' - \sin\theta\, y' \\ y = \sin\theta\, x' + \cos\theta\, y' \end{cases}$$

(b) $f(x,y) = f(x(x',y'), y(x',y'))$

$$\Rightarrow \frac{\partial f}{\partial x'} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial x'} = \cos\theta \frac{\partial f}{\partial x} + \sin\theta \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial y'} = -\sin\theta \frac{\partial f}{\partial x} + \cos\theta \frac{\partial f}{\partial y}$$

$$\Rightarrow \begin{cases} \frac{\partial}{\partial x'} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y'} = -\sin\theta \frac{\partial}{\partial x} + \cos\theta \frac{\partial}{\partial y} \end{cases} \text{ and } \begin{cases} \frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial x'} - \sin\theta \frac{\partial}{\partial y'} \\ \frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial x'} + \cos\theta \frac{\partial}{\partial y'} \end{cases}$$

(c) $M' = \cos\theta\, M + \sin\theta\, N$

$N' = -\sin\theta\, M + \cos\theta\, N$

(d) (i) $\frac{\partial N'}{\partial x'} - \frac{\partial M'}{\partial y'} = \left(\cos\theta \frac{\partial N'}{\partial x} + \sin\theta \frac{\partial N'}{\partial y}\right) - \left(-\sin\theta \frac{\partial M'}{\partial x} + \cos\theta \frac{\partial M'}{\partial y}\right)$

$= \cos\theta\left(-\sin\theta \frac{\partial M}{\partial x} + \cos\theta \frac{\partial N}{\partial x}\right) + \sin\theta\left(-\sin\theta \frac{\partial M}{\partial y} + \cos\theta \frac{\partial N}{\partial y}\right)$
$+ \sin\theta\left(\cos\theta \frac{\partial M}{\partial x} + \sin\theta \frac{\partial N}{\partial x}\right) - \cos\theta\left(\cos\theta \frac{\partial M}{\partial y} + \sin\theta \frac{\partial N}{\partial y}\right)$

$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

(ii) $\frac{\partial M'}{\partial x'} + \frac{\partial N'}{\partial y'} = \left(\cos\theta \frac{\partial M'}{\partial x} + \sin\theta \frac{\partial M'}{\partial y}\right) + \left(-\sin\theta \frac{\partial N'}{\partial x} + \cos\theta \frac{\partial N'}{\partial y}\right)$

$= \cos\theta\left(\cos\theta \frac{\partial M}{\partial x} + \sin\theta \frac{\partial N}{\partial x}\right) + \sin\theta\left(\cos\theta \frac{\partial M}{\partial y} + \sin\theta \frac{\partial N}{\partial y}\right)$
$-\sin\theta\left(-\sin\theta \frac{\partial M}{\partial x} + \cos\theta \frac{\partial N}{\partial x}\right) + \cos\theta\left(-\sin\theta \frac{\partial M}{\partial y} + \cos\theta \frac{\partial N}{\partial y}\right)$

$= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$

(e) $\begin{cases} \hat{x}' = (\cos\theta, \sin\theta) \\ \hat{y}' = (-\sin\theta, \cos\theta) \end{cases}$ in x-y coordinate

$\begin{cases} \hat{x} = (1, 0) \\ \hat{y} = (0, 1) \end{cases}$ in x-y coordinate

$\Rightarrow \begin{array}{l} \hat{x}' = \cos\theta\,\hat{x} + \sin\theta\,\hat{y} \\ \hat{y}' = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y} \end{array} \Rightarrow \begin{array}{l} \hat{x} = \cos\theta\,\hat{x}' - \sin\theta\,\hat{y}' \\ \hat{y} = \sin\theta\,\hat{x}' + \cos\theta\,\hat{y}' \end{array}$

(f) $M' = \cos\theta M + \sin\theta N$
$N' = -\sin\theta M + \cos\theta N$

(g) $\begin{vmatrix} n_1' & n_2' & n_3 \\ \partial_{x'} & \partial_{y'} & \partial_z \\ M' & N' & P \end{vmatrix} = \left(\frac{\partial P}{\partial y'} - \frac{\partial N'}{\partial z}\right)n_1' + \left(\frac{\partial M'}{\partial z} - \frac{\partial P}{\partial x'}\right)n_2' + \left(\frac{\partial N'}{\partial x'} - \frac{\partial M'}{\partial y'}\right)n_3$

$= \left(-\sin\theta \frac{\partial P}{\partial x} + \cos\theta \frac{\partial P}{\partial y} + \sin\theta \frac{\partial M}{\partial z} - \cos\theta \frac{\partial N}{\partial z}\right)(\cos\theta n_1 + \sin\theta n_2)$

$+ \left(\cos\theta \frac{\partial M}{\partial z} + \sin\theta \frac{\partial N}{\partial z} - \cos\theta \frac{\partial P}{\partial x} - \sin\theta \frac{\partial P}{\partial y}\right)(-\sin\theta n_1 + \cos\theta n_2)$

$$+\left(\cos\theta\frac{\partial N'}{\partial x}+\sin\theta\frac{\partial N'}{\partial y}+\sin\theta\frac{\partial M'}{\partial x}-\cos\theta\frac{\partial M'}{\partial y}\right)n_3$$

$$=\left(-\sin\theta\cos\theta\frac{\partial P}{\partial x}+\cos^2\theta\frac{\partial P}{\partial y}+\sin\theta\cos\theta\frac{\partial M}{\partial z}-\cos^2\theta\frac{\partial N}{\partial z}-\sin\theta\cos\theta\frac{\partial M}{\partial z}-\sin^2\theta\frac{\partial N}{\partial z}\right.$$
$$\left.+\sin\theta\cos\theta\frac{\partial P}{\partial x}+\sin^2\theta\frac{\partial P}{\partial y}\right)n_1$$

$$+\left(-\sin^2\theta\frac{\partial P}{\partial x}+\sin\theta\cos\theta\frac{\partial P}{\partial y}+\sin^2\theta\frac{\partial M}{\partial z}-\sin\theta\cos\theta\frac{\partial N}{\partial z}+\cos^2\theta\frac{\partial M}{\partial z}+\sin\theta\cos\theta\frac{\partial N}{\partial z}\right.$$
$$\left.-\cos^2\theta\frac{\partial P}{\partial x}-\sin\theta\cos\theta\frac{\partial P}{\partial y}\right)n_2$$

$$+\left(-\sin\theta\cos\theta\frac{\partial M}{\partial x}+\cos^2\theta\frac{\partial N}{\partial x}-\sin^2\theta\frac{\partial M}{\partial y}+\sin\theta\cos\theta\frac{\partial N}{\partial y}+\sin\theta\cos\theta\frac{\partial M}{\partial x}+\sin^2\theta\frac{\partial N}{\partial x}\right.$$
$$\left.-\cos^2\theta\frac{\partial M}{\partial y}-\sin\theta\cos\theta\frac{\partial N}{\partial y}\right)n_3$$

$$=\left(\frac{\partial P}{\partial y}-\frac{\partial N}{\partial z}\right)n_1+\left(\frac{\partial M}{\partial z}-\frac{\partial P}{\partial x}\right)n_2+\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)n_3$$

$$=\begin{vmatrix} n_1 & n_2 & n_3 \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix}$$

$$T_1'F_1'+T_2'F_2'+T_3F_3$$

$$=(\cos\theta T_1+\sin\theta T_2)(\cos\theta F_1+\sin\theta F_2)+(-\sin\theta T_1+\cos\theta T_2)(-\sin\theta F_1+\cos\theta F_2)+T_3F_3$$

$$=\cos^2\theta T_1F_1+\sin\theta\cos\theta T_1F_2+\sin\theta\cos\theta T_2F_1+\sin^2\theta T_2F_2+\sin^2\theta T_1F_1-\sin\theta\cos\theta T_1F_2$$
$$-\sin\theta\cos\theta T_2F_1+\cos^2\theta T_2F_2+T_3F_3$$

$$=T_1F_1+T_2F_2+T_3F_3$$