15-3. 20

Let $M = \frac{2x}{x^2+y^2+z^2}$, $N = \frac{2y}{x^2+y^2+z^2}$, $P = \frac{2z}{x^2+y^2+z^2}$

$\frac{\partial M}{\partial z} = \frac{-2xz}{(x^2+y^2+z^2)^2} = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial z} = \frac{-2yz}{(x^2+y^2+z^2)^2} = \frac{\partial P}{\partial y}$, $\frac{\partial M}{\partial y} = \frac{-2xy}{(x^2+y^2+z^2)} = \frac{\partial N}{\partial x}$

$\Rightarrow$ $M dx + N dy + P dz$ is exact $\Rightarrow$ F may be conservative

We try to find f s.t. $F = \nabla f$ i.e. $M = \frac{\partial f}{\partial x}$, $N = \frac{\partial f}{\partial y}$, $P = \frac{\partial f}{\partial z}$

for some g.

$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2+z^2} \Rightarrow f(x,y,z) = \ln(x^2+y^2+z^2) + g(y,z)$, $\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2+z^2} + \frac{\partial g}{\partial y} = N$

$\Rightarrow \frac{\partial g}{\partial y} = 0$, similarly , $\frac{\partial g}{\partial z} = 0 \Rightarrow f(x,y,z) = \ln(x^2+y^2+z^2) + C$, for some constant C

$\Rightarrow \int_{(-1,-1,-1)}^{(2,2,2)} M dx + N dy + P dz = f(2,2,2) - f(-1,-1,-1) = \ln(12) - \ln(3) = 2\ln 2$

24.

Let $M = \frac{x}{\sqrt{x^2+y^2+z^2}}$, $N = \frac{y}{\sqrt{x^2+y^2+z^2}}$, $P = \frac{z}{\sqrt{x^2+y^2+z^2}}$

$\frac{\partial M}{\partial z} = \frac{-xz}{(x^2+y^2+z^2)} = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial z} = \frac{-yz}{(x^2+y^2+z^2)} = \frac{\partial P}{\partial y}$, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow$ $M dx + N dy + P dz$ is exact $\Rightarrow$ F may be conservative

We try to find f s.t. $F = \nabla f$, i.e. $M = \frac{\partial f}{\partial x}$, $N = \frac{\partial f}{\partial y}$, $P = \frac{\partial f}{\partial z}$

$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} \Rightarrow f(x,y,z) = \sqrt{x^2+y^2+z^2} + g(y,z)$, for some g , $\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}} + \frac{\partial g}{\partial y} = N$

$\Rightarrow \frac{\partial g}{\partial y} = 0$, similarly , $\frac{\partial g}{\partial z} = 0 \Rightarrow f(x,y,z) = \sqrt{x^2+y^2+z^2} + C$, for some constant C

$\Rightarrow \int_A^B M dx + N dy + P dz = f(B) - f(A)$

30. $F = -GmM \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}$, Let $F = \nabla f$

$\frac{\partial f}{\partial x} = -GmM \frac{x}{(x^2+y^2+z^2)^{3/2}} \Rightarrow f(x,y,z) = \frac{GmM}{(x^2+y^2+z^2)^{1/2}} + g(y,z)$, for some function g

$-GmM \frac{y}{(x^2+y^2+z^2)^{3/2}} = \frac{\partial f}{\partial y} = \frac{GmM \cdot -y}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial g}{\partial y} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = g(z)$

$f(x,y,z) = \frac{GmM}{(x^2+y^2+z^2)^{1/2}} + C$

$-GmM \frac{z}{(x^2+y^2+z^2)^{3/2}} = \frac{\partial f}{\partial z} = \frac{GmM \cdot -z}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial g}{\partial z} \Rightarrow \frac{\partial g}{\partial z} = 0 \Rightarrow g(z) = C$, for some constant C

15.4 - 6.

$F = (x^2+4y)\,i + (x+y^2)\,j$. Let $M = x^2+4y$, $N = x+y^2$.

C: The square bounded by $x=0, x=1, y=0, y=1$

(I) counterclockwise circulation:

$$\oint_C F\cdot T\,ds = \int_0^1\int_0^1 \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_0^1\int_0^1 -3\,dx\,dy = -3.$$

(II) outward flux

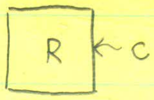
$$\oint_C F\cdot n\,ds = \int_0^1\int_0^1 \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\,dx\,dy$$

$$= \int_0^1\int_0^1 2x+2y\,dx\,dy = \int_0^1 x^2+2xy\Big|_0^1 dy$$

$$= \int_0^1 1+2y\,dy = y+y^2\Big|_0^1 = 2$$

22.

Let $xy^2 = M$, $x^2y+2x = N$,



then $$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint_R (2+2xy)-(2xy)\,dx\,dy$$

$$= \iint_R 2\,dx\,dy$$

$$= 2\times(\text{the area of the square})$$

26.

$\because F$ is conservative , $\therefore \exists f$ s.t. $F = \nabla f$

For $c: \vec{r}(t)$, $a\le t\le b$, $\vec{r}(a) = \vec{r}(b)$, then $\oint F\cdot T\,ds = f(\vec{r}(b)) - f(\vec{r}(a)) = 0$

Moreover, By Green's Thm tangential form

$\oint_C F\cdot T\,ds = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dx\,dy = \iint_R 0\,dxdy = 0 \Rightarrow$ We can get same result.