Homework for week 9

13.6-54

a)

$A = \vec{i} + \vec{j} - k \Rightarrow |A| = \sqrt{3}$,

Let $u = \frac{A}{|A|} = \frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} - \frac{1}{\sqrt{3}}k$, then $D_u f|_p := 2\sqrt{3}$

$\Rightarrow f_x(p) \cdot \frac{1}{\sqrt{3}} + f_y(p) \cdot \frac{1}{\sqrt{3}} - f_z(p) \cdot \frac{1}{\sqrt{3}} = 2\sqrt{3}$

$\Rightarrow f_x(p) + f_y(p) - f_z(p) = 6$

$\because D_u f|_p$ is greatest $\therefore \nabla f_p \parallel u$

$\Rightarrow \nabla f|_p = cu = \frac{c}{\sqrt{3}}(\vec{i} + \vec{j} - k)$, for some constant c

$\Rightarrow \frac{c}{\sqrt{3}}(1+1-1) = 6$

$\Rightarrow c = 6\sqrt{3}$. Hence $\nabla f|_p = 6\vec{i} + 6\vec{j} - 6k$

b)

Let $A = \vec{i} + \vec{j}$, $|A| = \frac{1}{\sqrt{2}}$, Let $u = \frac{1}{\sqrt{2}}(\vec{i} + \vec{j})$

$D_u f|_p = f_x(p) \cdot \frac{1}{\sqrt{2}} + f_y(p) \cdot \frac{1}{\sqrt{2}} + f_z(p) \cdot 0$

$= 6 \cdot \frac{1}{\sqrt{2}} + 6 \cdot \frac{1}{\sqrt{2}} = 6\sqrt{2}$

57.

a) $r = \sqrt{t}\,i + \sqrt{t}\,j - \frac{1}{4}(t+3)k$,

the tangent line of r is $T = \frac{1}{2\sqrt{t}}\,i + \frac{1}{2\sqrt{t}}\,j - \frac{1}{4}k$

$t=1 \Rightarrow P_0 = r|_{t=1} = (1,1,-1)$, $T|_{t=1} = \frac{1}{2}i + \frac{1}{2}j - \frac{1}{4}k$

$f(x,y,z) = x^2+y^2-z-3$ $\quad \nabla f|_{P_0} = 2xi+2yj-k|_{P_0}$

$= 2i+2j-k$

$= 4(\frac{1}{2}i+\frac{1}{2}j-\frac{1}{4}k)$

$= 4\,T|_{t=1}$

$\Rightarrow$ r is normal to the f when $t=1$.

b) $r = \sqrt{t}\,i + \sqrt{t}\,j + (2t-1)k$

the tangent line of r is $T: \frac{1}{2\sqrt{t}}\,t + \frac{1}{2\sqrt{t}}\,j + 2k$

$t=1 \Rightarrow P_0 = r|_{t=1} = (1,1,1)$, $T|_{t=1} = \frac{1}{2}i + \frac{1}{2}j + 2k$

$f(x,y,z) = x^2+y^2-z-1$, $\nabla f|_{P_0} = 2xi+2yj-k\,|_{P_0}$

$= 2i+2j-k$

$T \cdot \nabla f = (\frac{1}{2}i+\frac{1}{2}j+2k)\cdot(2i+2j-k)$

$= 1+1-2$

$= 0$

$\Rightarrow$ r is tangent to f when $t=1$

58.

a)

$$\nabla(kf) = \frac{\partial kf}{\partial x} i + \frac{\partial kf}{\partial y} j + \frac{\partial kf}{\partial z} k$$

$$= k\left(\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k\right) = k \nabla f$$

b) c)

$$\nabla(f \pm g) = \frac{\partial(f\pm g)}{\partial x} i + \frac{\partial(f\pm g)}{\partial y} j + \frac{\partial(f\pm g)}{\partial z} k$$

$$= \left(\frac{\partial f}{\partial x} \pm \frac{\partial g}{\partial x}\right) i + \left(\frac{\partial f}{\partial y} \pm \frac{\partial g}{\partial y}\right) j + \left(\frac{\partial f}{\partial z} \pm \frac{\partial g}{\partial z}\right) k$$

$$= \left(\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k\right) \pm \left(\frac{\partial g}{\partial x} i + \frac{\partial g}{\partial y} j + \frac{\partial g}{\partial z} k\right)$$

$$= \nabla f \pm \nabla g$$

d)

$$\nabla(fg) = \frac{\partial fg}{\partial x} i + \frac{\partial fg}{\partial y} j + \frac{\partial fg}{\partial z} k$$

$$= \left(\frac{\partial f}{\partial x} g + f \frac{\partial g}{\partial x}\right) i + \left(\frac{\partial f}{\partial y} g + f \frac{\partial g}{\partial y}\right) j + \left(\frac{\partial f}{\partial z} g + f \frac{\partial g}{\partial z}\right) k$$

$$= g\left(\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k\right) + f\left(\frac{\partial g}{\partial x} i + \frac{\partial g}{\partial y} j + \frac{\partial g}{\partial z} k\right)$$

$$= g \nabla f + f \nabla g$$

e)

$$\nabla\left(\frac{f}{g}\right) = \frac{\partial\left(\frac{f}{g}\right)}{\partial x}\,i + \frac{\partial\left(\frac{f}{g}\right)}{\partial y}\,j + \frac{\partial\left(\frac{f}{g}\right)}{\partial z}\,k$$

$$= \left(\frac{\frac{\partial f}{\partial x}g - f\frac{\partial g}{\partial x}}{g^2}\right) i + \left(\frac{\frac{\partial f}{\partial y}g - f\frac{\partial g}{\partial y}}{g^2}\right) j + \left(\frac{\frac{\partial f}{\partial z}g - f\frac{\partial g}{\partial z}}{g^2}\right) k$$

$$= \frac{1}{g^2}\left[g\left(\frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k\right) - f\left(\frac{\partial g}{\partial x}i + \frac{\partial g}{\partial y}j + \frac{\partial g}{\partial z}k\right)\right]$$

$$= \frac{1}{g^2}\left(g\nabla f - f\nabla g\right)$$

S13.5 - extra 3

Let $\mathbf{x} = (x, y, z)$ and $\mathbf{x}_0 = (x_0, y_0, z_0)$

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \frac{1}{2}D^2 f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0, \mathbf{x} - \mathbf{x}_0) + Re(\mathbf{x})$$

$$= f(\mathbf{x}_0) + \left[f_x(\mathbf{x}_0)(x - x_0) + f_y(\mathbf{x}_0)(y - y_0) + f_z(\mathbf{x}_0)(z - z_0)\right]$$
$$+ \frac{1}{2}\left[f_{xx}(\mathbf{x}_0)(x - x_0)^2 + f_{yy}(\mathbf{x}_0)(y - y_0)^2 + f_{zz}(\mathbf{x}_0)(z - z_0)^2\right.$$
$$\left. + 2f_{xy}(\mathbf{x}_0)(x - x_0)(y - y_0) + 2f_{xz}(\mathbf{x}_0)(x - x_0)(z - z_0) + 2f_{yz}(\mathbf{x}_0)(y - y_0)(z - z_0)\right] + Re(\mathbf{x})$$

$$Re(\mathbf{x}) = \frac{1}{3!}\left[f_{xxx}(c)(x - x_0)^3 + f_{yyy}(c)(y - y_0)^3 + f_{zzz}(c)(z - z_0)^3\right.$$
$$+ 3f_{xxy}(c)(x - x_0)^2(y - y_0) + 3f_{xyy}(c)(x - x_0)(y - y_0)^2$$
$$+ 3f_{xxz}(c)(x - x_0)^2(z - z_0) + 3f_{xzz}(c)(x - x_0)(z - z_0)^2$$
$$+ 3f_{yyz}(c)(y - y_0)^2(z - z_0) + 3f_{yzz}(c)(y - y_0)(z - z_0)^2$$
$$\left. + 6f_{xyz}(c)(x - x_0)(y - y_0)(z - z_0)\right]$$

, for $c = (c_1, c_2, c_3)$, and c is on the line segment between $\mathbf{x}$ and $\mathbf{x}_0$