13.4-8 $f(x,y)=\frac{1}{2}x^2+xy+\frac{1}{4}y^2+3x-3y+4$, $P_0=(2,2)$

$$\Rightarrow \begin{cases} f_x(x,y)=x+y+3 \\ f_y(x,y)=x+\frac{1}{2}y-3 \end{cases} \Rightarrow \begin{cases} f_x(2,2)=7 \\ f_y(2,2)=0 \end{cases}$$

$$\Rightarrow L(x,y)=f(2,2)+f_x(2,2)\cdot(x-2)+f_y(2,2)\cdot(y-2)$$
$$=11+7(x-2)=7x-3$$

Since $|E|\le \frac{M}{2}(|x-2|+|y-2|)^2$, where $M=\max\limits_{\substack{|x-2|\le 0.1\\|y-2|\le 0.1}}\{|f_{xx}(x,y)|,|f_{xy}(x,y)|,|f_{yx}(x,y)|,|f_{yy}(x,y)|\}$

and $f_{xx}(x,y)=1$, $f_{xy}(x,y)=1$, $f_{yx}(x,y)=1$, $f_{yy}(x,y)=\frac{1}{2}$

$\Rightarrow M=1$

$\Rightarrow$ When $|x-2|\le 0.1$, $|y-2|\le 0.1$, $|E|\le\frac{1}{2}(0.1+0.1)^2=0.02$

36. If f has continuous second partial derivatives on an open region R

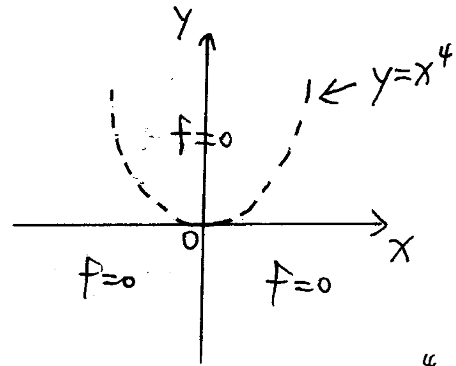
$\Rightarrow f_{xx}$, f_{xy}, f_{yx} , f_{yy} are continuous on R

By P824 corollary of theorem 3 ,

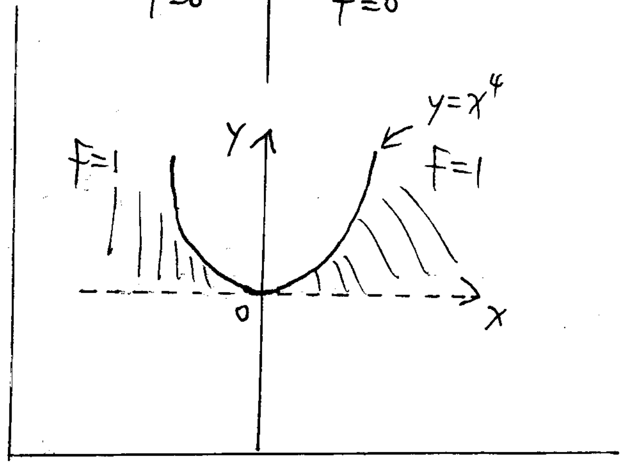
f_{xx}, f_{xy} are continuous on R $\Rightarrow f_x$ is differentiable on R
$\Rightarrow f_x$ is continuous on R

f_{yx} , f_{yy} are continuous on R $\Rightarrow f_y$ is differentiable on R
$\Rightarrow f_y$ is continuous on R

5.13.4-extra) It doesn't mean that $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$



Let $f(x,y) = \begin{cases} 0, & y > x^4 \\ 0, & y \leq 0 \\ 1, & \text{others} \end{cases}$



From ①, we have

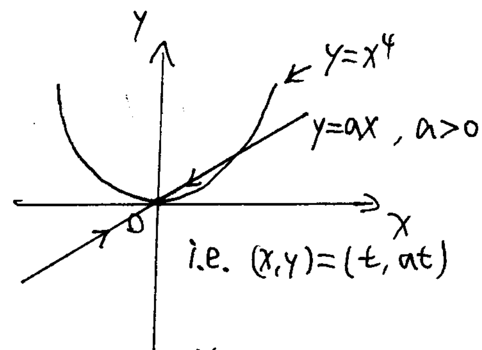
$$\lim_{t\to 0} f(t, at) = \lim_{t\to 0} f(at, t) = 0 = f(0,0)$$

From ②, we have

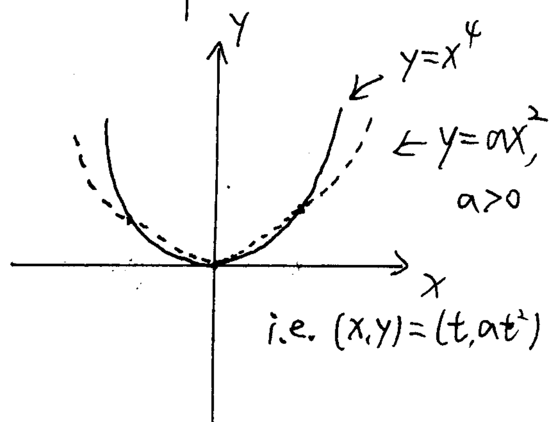
$$\lim_{t\to 0} f(t, at^2) = \lim_{t\to 0} f(at^2, t) = 0 = f(0,0)$$

but $\lim_{\substack{y=x^4 \\ x\to 0}} f(x,y) = \lim_{x\to 0} 1 = 1 \neq f(0,0)$

①



②



5|3.4-extra2 (i) $\because \sqrt{x}$ is continuous on $x \geq 0$ and x^2+2y^2 is continuous on $\mathbb{R}^2$

$\therefore \sqrt{x^2+2y^2}$ is continuous on $\mathbb{R}^2$

(ii) $\because f_x(x,y) = \dfrac{x}{\sqrt{x^2+2y^2}} \Rightarrow f_x(x,y)$ does not exist at $(0,0)$

$\Rightarrow f$ is not differentiable at $(0,0)$

§13.4-extra3 (i) If $f(x,y)=o(1)\cdot|x-x_0|+o(1)\cdot|y-y_0|$ as $(x,y)\to(x_0,y_0)$

$$\Rightarrow f(x,y)=g(x,y)\,|x-x_0|+h(x,y)\,|y-y_0|$$

where $\lim\limits_{(x,y)\to(x_0,y_0)}\dfrac{g(x,y)}{1}=0$ and $\lim\limits_{(x,y)\to(x_0,y_0)}\dfrac{h(x,y)}{1}=0$

$$\Rightarrow f(x,y)=\left(g(x,y)\frac{|x-x_0|}{\sqrt{(x-x_0)^2+(y-y_0)^2}}+h(x,y)\frac{|y-y_0|}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\right)\sqrt{(x-x_0)^2+(y-y_0)^2}$$

$$\because \frac{|x-x_0|}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\le 1 \Rightarrow \left|g(x,y)\frac{|x-x_0|}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\right|\le|g(x,y)|$$

and $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y)=0$

By Sandwich theorem, $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y)\dfrac{|x-x_0|}{\sqrt{(x-x_0)^2+(y-y_0)^2}}=0$

Similarly, we have $\lim\limits_{(x,y)\to(x_0,y_0)} h(x,y)\dfrac{|y-y_0|}{\sqrt{(x-x_0)^2+(y-y_0)^2}}=0$

hence $\lim\limits_{(x,y)\to(x_0,y_0)}\dfrac{g(x,y)\frac{|x-x_0|}{\sqrt{(x-x_0)^2+(y-y_0)^2}}+h(x,y)\frac{|y-y_0|}{\sqrt{(x-x_0)^2+(y-y_0)^2}}}{1}=0$

$\Rightarrow f(x,y)=o(1)\cdot\sqrt{(x-x_0)^2+(y-y_0)^2}$ as $(x,y)\to(x_0,y_0)$

(ii) If $f(x,y)=o(1)\cdot\sqrt{(x-x_0)^2+(y-y_0)^2}$ as $(x,y)\to(x_0,y_0)$

$\Rightarrow f(x,y)=k(x,y)\cdot\sqrt{(x-x_0)^2+(y-y_0)^2}$ where $\lim\limits_{(x,y)\to(x_0,y_0)}\dfrac{k(x,y)}{1}=0$

$$\Rightarrow f(x,y)=\left(k(x,y)\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{|x-x_0|+|y-y_0|}\right)\left(|x-x_0|+|y-y_0|\right)$$

$\because \dfrac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{|x-x_0|+|y-y_0|}\le 1 \Rightarrow \left|k(x,y)\dfrac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{|x-x_0|+|y-y_0|}\right|\le|k(x,y)|$ and $\lim\limits_{(x,y)\to(x_0,y_0)} k(x,y)=0$

By Sandwich theorem, $\lim\limits_{(x,y)\to(x_0,y_0)} k(x,y)\dfrac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{|x-x_0|+|y-y_0|}=0$

$$\Rightarrow f(x,y) = o(1) \cdot \left(|x-x_0| + |y-y_0| \right) = o(1)\,|x-x_0| + o(1)\,|y-y_0|$$

5/3.5-extra 1 (i) $\frac{d}{dy}\int_1^2 \frac{\cos(xy)}{x}\,dx = \int_1^2 \frac{d}{dy}\left(\frac{\cos(xy)}{x}\right)dx = \int_1^2 -\sin(xy)\,dx = \frac{1}{y}\cos(xy)\Big|_1^2$

$$= \frac{1}{y}(\cos(2y) - \cos y)$$

(ii) Let $a(y) = 1+y^2$, $b(y) = 2+\sin y$, $k(y) = y$

$$\Rightarrow \int_{1+y^2}^{2+\sin y} \frac{\cos(xy)}{x}\,dx = \int_{a(y)}^{b(y)} \frac{\cos(xk(y))}{x}\,dx \overset{\text{let}}{=} F(a(y), b(y), k(y))$$

$$\Rightarrow \frac{d}{dy}\int_{1+y^2}^{2+\sin y} \frac{\cos(xy)}{x}\,dx = \frac{d}{dy} F(a(y), b(y), k(y))$$

$$= \frac{\partial F}{\partial a}\cdot\frac{da}{dy} + \frac{\partial F}{\partial b}\cdot\frac{db}{dy} + \frac{\partial F}{\partial k}\cdot\frac{dk}{dy}$$

Since $\frac{\partial F}{\partial a} = \frac{\partial}{\partial a}\int_a^b \frac{\cos(xk)}{x}\,dx = -\frac{\partial}{\partial a}\left(\int_b^a \frac{\cos(xk)}{x}\,dx\right) = -\frac{\cos(ak)}{a}$

$$\frac{\partial F}{\partial b} = \frac{\partial}{\partial b}\int_a^b \frac{\cos(xk)}{x}\,dx = \frac{\cos(bk)}{b}$$

$$\frac{\partial F}{\partial k} = \frac{\partial}{\partial k}\int_a^b \frac{\cos(xk)}{x}\,dx = \int_a^b \frac{\partial}{\partial k}\left(\frac{\cos(xk)}{x}\right)dx = \int_a^b -\sin(xk)\,dx$$

$$= \frac{1}{k}(\cos(bk) - \cos(ak))$$

$$\Rightarrow \frac{d}{dy}\int_{1+y^2}^{2+\sin y} \frac{\cos(xy)}{x}\,dx = -\frac{\cos((1+y^2)y)}{1+y^2}\cdot(2y) + \frac{\cos((2+\sin y)y)}{2+\sin y}\cdot\cos y + \frac{1}{y}\left(\cos((2+\sin y)y) - \cos((1+y^2)y)\right)$$