Homework for week 6 ~ week 7.

10.8 - 22.

the area $= \int_0^{2\pi} \frac{1}{2}(\cos\theta+1)^2 d\theta - \int_0^{\pi} \frac{1}{2}\cos^2\theta \, d\theta$

$= \frac{1}{2}\int_0^{2\pi} (\cos\theta+1)^2 - \cos^2\theta \, d\theta + \frac{1}{2}\int_{\pi}^{2\pi} \cos^2\theta \, d\theta$

$= \pi + \frac{1}{4}\int_{\pi}^{2\pi} 1+\cos 2\theta \, d\theta$

$= \pi + \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi}^{2\pi}$

$= \frac{5}{4}\pi$

35.

$r = f(\theta) \Rightarrow \frac{dr}{d\theta} = f'(\theta)$, $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$,

$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$

$= (f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2$

$= (f'(\theta)\cos\theta)^2 + (f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta)^2 + (f(\theta)\cos\theta)^2$

$= (f'(\theta))^2 + (f(\theta))^2$

$= \left(\frac{dr}{d\theta}\right)^2 + r^2$

$\Rightarrow \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$

37.

Let $r_1 = 2f(\theta)$, $r_2 = f(\theta)$

Let L_i is the length of curve r_i and $A_i(B_i)$ is the area of the surface generated by revolving the curve r_i about x-axis (y-axis), $i=1,2$

a) $$L_1 = \int_\alpha^\beta \sqrt{r_1^2 + \left(\frac{dr_1}{d\theta}\right)^2}\, d\theta = \int_\alpha^\beta \sqrt{4f(\theta)^2 + 4f'(\theta)^2}\, d\theta$$

$$= 2\int_\alpha^\beta \sqrt{f(\theta)^2 + (f'(\theta))^2}\, d\theta$$

$$= 2\int_\alpha^\beta \sqrt{r_2^2 + \left(\frac{dr_2}{d\theta}\right)^2}\, d\theta = 2L_2$$

b) $$A_1 = \int_\alpha^\beta 2\pi r_1 \sin\theta \sqrt{r_1^2 + \left(\frac{dr_1}{d\theta}\right)^2}\, d\theta$$

$$= \int_\alpha^\beta 2\pi \cdot 2f(\theta) \sin\theta\, 2\sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$$

$$= 4\int_\alpha^\beta 2\pi f(\theta) \sin\theta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$$

$$= 4\int_\alpha^\beta 2\pi r_2 \sin\theta \sqrt{r_2^2 + \left(\frac{dr_2}{d\theta}\right)^2}\, d\theta = 4A_2$$

Similarly, $B_1 = 4B_2$

13.2-34.

a) $h(x,y,z) = \frac{1}{|y|+|z|}$ is continuous except x-axis

b) $h(x,y,z) = \frac{1}{|xy|+|z|}$ is continuous except x-axis and y-axis

45.

$1-\frac{x^2y^2}{3} < \frac{\tan^{-1}xy}{xy} < 1$

$\because \lim_{(x,y)\to(0,0)} 1-\frac{x^2y}{3} = \lim_{(x,y)\to(0,0)} 1 = 1$ (since $1-\frac{x^2y}{3}$ is continuous on $\mathbb{R}^2$)

$\therefore \lim_{(x,y)\to(0,0)} \frac{\tan^{-1}xy}{xy} = 1.$

47.

$-1 \le \sin\left(\frac{1}{x}\right) \le 1 \Rightarrow -y \le y\sin\left(\frac{1}{x}\right) \le y$

$\because \lim_{(x,y)\to(0,0)} y = \lim_{(x,y)\to(0,0)} -y = 0$ (since y and $-y$ are continuous on $\mathbb{R}^2$)

$\therefore \lim_{(x,y)\to(0,0)} y\sin\left(\frac{1}{x}\right) = 0.$