9.8-28 $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$, $|x|<1$

Let $P_n(x) = x - \frac{x^2}{2} + \cdots + (-1)^{n-1} \frac{x^n}{n}$

$\Rightarrow |\ln(1.1) - P_n(1.1)| \le \frac{(0.1)^{n+1}}{n+1}$ (Alternating Series Estimation Theorem)

Solve $\frac{(0.1)^{n+1}}{n+1} \le 10^{-8} \Rightarrow n \ge 7$

$\Rightarrow |\ln(1.1) - P_7(1.1)| \le \frac{(0.1)^8}{8} \le 10^{-8}$

31. $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)}$, $|x|<\frac{\pi}{2}$

Let $P_{2n-1}(x) = x - \frac{x^3}{3} + \cdots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1}$

$\Rightarrow |\tan^{-1}x - P_{2n-1}(x)| \le \frac{|x|^{2n+1}}{(2n+1)}$, $|x|<1$ (Alternating Series Estimation Theorem)

Solve $\frac{(\frac{1}{18})^{2n+1}}{(2n+1)} \cdot 48 \le \frac{1}{3} \cdot 10^{-6}$

$\Rightarrow (18)^{2n+1} \cdot (2n+1) \ge 10^6 \cdot 144 \Rightarrow (2n+1)\log_{10}18 + \log_{10}(2n+1) \ge 6 + \log_{10}144$

$\Rightarrow (2n+1)(1.2552) + \log_{10}(2n+1) \ge 6 + 2.1582 = 8.1582$

$\Rightarrow n \ge 3$

32 The first three nonzero terms of the Maclaurin series for $\tan t$

is $t + \frac{t^3}{3} + \frac{2}{15}t^5$

$\Rightarrow$ For $\ln \sec x$ is $\ln \sec 0 + \int_0^x t + \frac{t^3}{3} + \frac{2}{15}t^5 \, dt = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45}$

34. (a) The first four nonzero terms of the Maclaurin series for

$\frac{1}{\sqrt{1+t^2}} = (1+t^2)^{-\frac{1}{2}}$ is $1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 - \frac{5}{16}t^6$

$\Rightarrow$ For $\sinh^{-1} x$ is $\int_0^x 1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 - \frac{5}{16}t^6 \, dt = x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7$

Ch9-28 $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges $\Leftrightarrow p > 1$

Hence $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges absolutely ($p=2$)

30 $\sum_{n=3}^{\infty} \frac{\ln n}{\ln(\ln n)}$

$\because \lim_{n\to\infty} \frac{\ln n}{\ln(\ln n)} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{\ln n}\cdot\frac{1}{n}} = \lim_{n\to\infty} \ln n = \infty$

$\therefore$ The series diverges.

32. $\sum_{n=1}^{\infty} \frac{(-1)^n 3n^2}{n^3+1}$

$\because \lim_{n\to\infty} \frac{\frac{3n^2}{n^3+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{3n^3}{n^3+1} = 3$, and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$\therefore$ By limit Comparison test , $\sum_{n=1}^{\infty} \frac{3n^2}{n^3+1}$ diverges

Moreover, $\left(\frac{3x^2}{x^3+1}\right)' = \frac{3x(2-x^3)}{(x^3+1)^2} < 0$ when $x \geq 2$

and $\frac{3n^2}{n^3+1} > 0, \forall n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{3n^2}{n^3+1} = 0$

By Alternating Series theorem , $\sum_{n=1}^{\infty} \frac{(-1)^n 3n^2}{n^3+1}$ converges

$\Rightarrow$ The series converges conditionally

49. $$\frac{1}{1+x} = 1-x+x^2-x^3+\cdots = \sum_{n=0}^{\infty}(-x)^n \quad , |x|<1$$

$$\Rightarrow 1-\frac{1}{4}+\frac{1}{16}-\cdots = \sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n = \frac{1}{1+\frac{1}{4}} = \frac{4}{5}$$

50 $$\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n} \quad , |x|<1$$

$$\Rightarrow \frac{2}{3}-\frac{4}{18}+\frac{8}{81}-\cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{2}{3}\right)^n = \ln\left(1+\frac{2}{3}\right) = \ln\frac{5}{3}$$

51. $$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad , x\in\mathbb{R}$$

$$\Rightarrow \pi-\frac{\pi^3}{3!}+\frac{\pi^5}{5!}-\cdots = \sum_{n=0}^{\infty}\frac{(-1)^n \pi^{2n+1}}{(2n+1)!} = \sin\pi = 0$$

52. $$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} \quad , x\in\mathbb{R}$$

$$\Rightarrow 1-\frac{\pi^2}{9\cdot 2!}+\frac{\pi^4}{81\cdot 4!}-\cdots = \sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\pi}{3}\right)^{2n}}{(2n)!} = \cos\frac{\pi}{3} = \frac{1}{2}$$

53. $$e^x = 1+x+\frac{x^2}{2!}+\cdots = \sum_{n=0}^{\infty}\frac{x^n}{n!} \quad , x\in\mathbb{R}$$

$$\Rightarrow 1+\ln 2+\frac{(\ln 2)^2}{2!}+\cdots = \sum_{n=0}^{\infty}\frac{(\ln 2)^n}{n!} = e^{\ln 2} = 2$$

54. $$\tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)} \quad , -\frac{\pi}{2}<x<\frac{\pi}{2}$$

$$\Rightarrow \frac{1}{\sqrt{3}}-\frac{1}{9\sqrt{3}}+\cdots = \sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{1}{\sqrt{3}}\right)^{2n+1}}{(2n+1)} = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

71. $\because \sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0 \Rightarrow \exists k\in\mathbb{N}$ s.t. $0\le a_n<1$, $\forall n\ge k$

$$\Rightarrow \sum_{n=k}^{\infty} a_n b_n \le \sum_{n=k}^{\infty} b_n$$

$\because \sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=k}^{\infty} a_n b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n b_n$ converges

72 (i) Let $a_n = b_n = \frac{1}{n}$ $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both diverge

But $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

(ii) Let $a_n = b_n = 1$ $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both diverge

But $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} 1$ diverges

By (i), (ii), we know nothing about $\sum_{n=1}^{\infty} a_n b_n$

§9.8-extra 2

$$\begin{cases} \frac{dy}{dx} = 1+y^2 \\ y(0)=0 \end{cases}$$

(i) Assume $y(x) = y_0 + y_1 x + y_2 x^2 + \cdots = \sum_{n=0}^{\infty} y_n x^n$, where y_n, $n=0,1,2,\cdots$ are constants

$y(0)=0 \Rightarrow y_0 = 0$

$$\frac{dy}{dx} = y_1 + 2y_2 x + 3y_3 x^2 + \cdots = \sum_{n=1}^{\infty} n y_n x^{n-1}$$

$$\parallel$$

$$1+y^2 = 1 + y_0^2 + (2y_0 y_1)x + (y_1^2 + 2y_0 y_2)x^2 + \cdots$$

constant : $y_1 = 1 + y_0^2 = 1$

x^1 : $2y_2 = 2y_0 y_1 = 0 \Rightarrow y_2 = 0$

x^2 : $3y_3 = y_1^2 + 2y_0 y_2 = 1 \Rightarrow y_3 = \frac{1}{3}$

$\Rightarrow y(x) = x + \frac{x^3}{3} + \cdots$

(ii) $\because \frac{dy}{dx} = 1+y^2$

$\therefore \int \frac{1}{1+y^2}\,dy = \int 1\,dx$

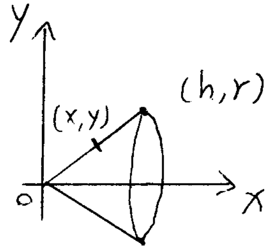
$\Rightarrow \tan^{-1} y = x + C$

$y(0)=0 \Rightarrow \tan^{-1} 0 = 0 + C \Rightarrow \therefore C = 0$

$\Rightarrow \tan^{-1} y = x \Rightarrow y = \tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots$

By (i), (ii), the first two nonzero coefficients of the Taylor series expansion of the exact solution match y_1, y_3

10.4-28 $\begin{cases} x = ht \\ y = rt \end{cases} \quad 0 \le t \le 1$



$$\text{Surface area} = \int_0^1 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^1 2\pi (rt) \sqrt{h^2 + r^2}\, dt$$

$$= 2\pi r \sqrt{h^2 + r^2} \int_0^1 t\, dt = \pi r \sqrt{h^2 + r^2}$$

10.6-15 $r = -1 + \cos\theta \;, \quad \theta = \pm \frac{\pi}{2}$

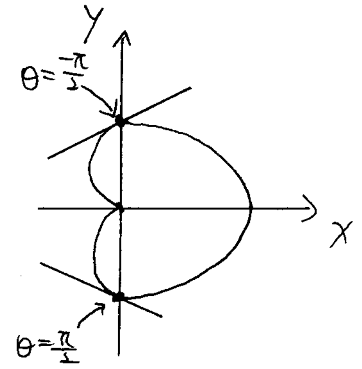
$$\begin{cases} x = r\cos\theta = (-1 + \cos\theta)\cos\theta = -\cos\theta + \cos^2\theta \\ y = r\sin\theta = (-1 + \cos\theta)\sin\theta = -\sin\theta + \sin\theta\cos\theta \end{cases}$$

$$\Rightarrow \begin{cases} \frac{dx}{d\theta} = \sin\theta + 2\cos\theta(-\sin\theta) = \sin\theta - 2\sin\theta\cos\theta \\ \frac{dy}{d\theta} = -\cos\theta + \cos^2\theta - \sin^2\theta \end{cases}$$

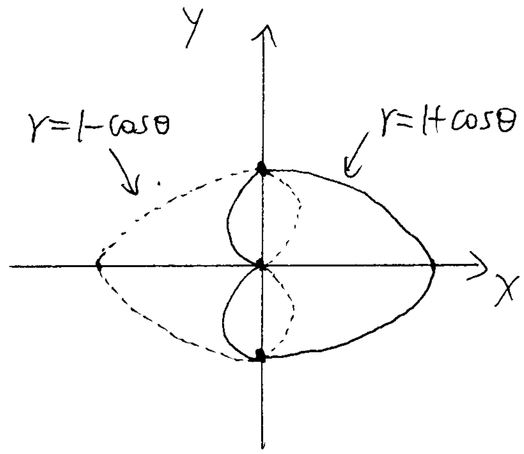
$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-\cos\theta + \cos^2\theta - \sin^2\theta}{\sin\theta - 2\sin\theta\cos\theta}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}} = \frac{-0+0-1}{1-2\cdot 1\cdot 0} = \frac{-1}{1} = -1$$

and $$\left.\frac{dy}{dx}\right|_{\theta=\frac{-\pi}{2}} = \frac{-0+0-1}{-1-2\cdot(-1)\cdot 0} = \frac{-1}{-1} = 1$$



31. $$\begin{cases} r = 1+\cos\theta \\ r = 1-\cos\theta \end{cases}$$



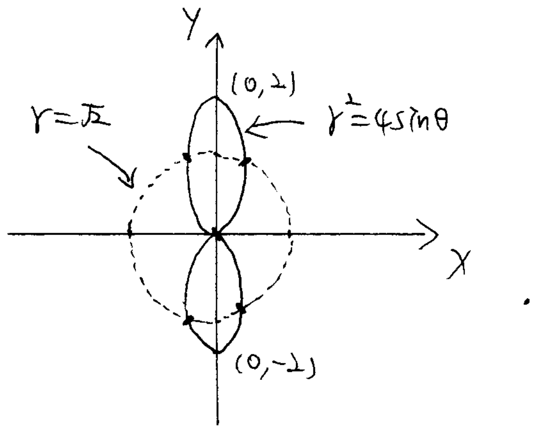
$$1+\cos\theta = r = 1-\cos\theta$$

$$\Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\Rightarrow \left(1, \frac{\pi}{2}\right), \left(1, \frac{3\pi}{2}\right)$$

Points of intersection : $\left(1, \frac{\pi}{2}\right), \left(1, \frac{3\pi}{2}\right)$, $(0,0)$ from graph

35. $$\begin{cases} r = \sqrt{2} \\ r^2 = 4\sin\theta \end{cases}$$



$$2 = r^2 = 4\sin\theta$$

$$\Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5}{6}\pi$$

$$\Rightarrow \left(\pm\sqrt{2}, \frac{\pi}{6}\right), \left(\pm\sqrt{2}, \frac{5}{6}\pi\right)$$

Points of intersection : $\left(\sqrt{2}, \frac{\pi}{6}\right)$, $\left(\sqrt{2}, \frac{-\pi}{6}\right)$, $\left(\sqrt{2}, \frac{5}{6}\pi\right)$, $\left(\sqrt{2}, \frac{-5}{6}\pi\right)$

$\left(\sqrt{2}, \frac{-\pi}{6}\right) = \left(-\sqrt{2}, \frac{5}{6}\pi\right)$, $\left(\sqrt{2}, \frac{-5}{6}\pi\right) = \left(-\sqrt{2}, \frac{\pi}{6}\right)$