9.7-33 By Taylor's formula

$$e^x = e^a + \frac{e^a}{1!}(x-a) + \frac{e^a}{2!}(x-a)^2 + \cdots + \frac{e^a}{n!}(x-a)^n + \frac{e^c}{(n+1)!}(x-a)^{n+1}$$

where c is between a and x

Since $e^c \le \max\{e^a, e^x\}$ and $\lim_{n\to\infty} \frac{(x-a)^{n+1}}{(n+1)!} = 0$, $\forall x \in \mathbb{R}$

$$\Rightarrow \lim_{n\to\infty} \frac{e^c}{(n+1)!}(x-a)^{n+1} = 0$$

$$\Rightarrow e^x = \sum_{n=0}^{\infty} e^a \frac{(x-a)^n}{n!} = e^a\left[1 + (x-a) + \frac{1}{2!}(x-a)^2 + \cdots\right]$$

47(a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$$\Rightarrow \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots$$

$$\Rightarrow \frac{\sin x}{x} = 1 - \left(\frac{x^2}{3!} - \frac{x^4}{5!}\right) - \left(\frac{x^6}{7!} - \frac{x^8}{9!}\right) - \cdots < 1 \quad \text{when } 0<|x|\le 4$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \left(\frac{x^4}{5!} - \frac{x^6}{7!}\right) + \left(\frac{x^8}{9!} - \frac{x^{10}}{11!}\right) + \cdots > 1 - \frac{x^2}{3!} \quad \text{when } 0<|x|\le 4$$

$$\Rightarrow 1 - \frac{x^2}{6} < \frac{\sin x}{x} < 1 \quad \text{when } 0<|x|\le 4$$

when $|x|>4$, $\left|\frac{\sin x}{x}\right| \le \frac{1}{4}$ and $1-\frac{x^2}{6} < 1 - \frac{4^2}{6} = -\frac{5}{3} < -\frac{1}{4}$

$$\Rightarrow 1 - \frac{x^2}{6} < -\frac{1}{4} \le \frac{\sin x}{x} \le \frac{1}{4} < 1 \quad \text{when } |x|>4$$

$$\Rightarrow 1 - \frac{x^2}{6} < \frac{\sin x}{x} < 1, \quad x \ne 0$$

50 (a) $\because f^{(n)}(0)=0, \forall n \in \mathbb{N}$

$\therefore \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = 0$

The Maclaurin series generated by f is 0, and 0

converges for all $x \in \mathbb{R}$

But 0 converges to $f(x)$ only at $x=0$

(b) Taylor's formula:

$$f(x) = 0 + \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \quad \text{for some } c \text{ between } 0 \text{ and } x$$

$$\Rightarrow \lim_{n\to\infty} \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} = f(x)$$

$$\because f(x)=0 \Leftrightarrow x=0$$

$$\therefore \lim_{n\to\infty} \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} = 0 \Leftrightarrow x=0$$

$\Rightarrow$ 0 converges to $f(x)$ only at $x=0$

57 $f(x)=\sum_{n=0}^{\infty} a_n x^n$ on $(-c, c)$

$$\Rightarrow f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1) a_n x^{n-k} \quad \text{on } (-c, c)$$

$$\Rightarrow f^{(k)}(0) = k!\, a_k$$

$\Rightarrow$ Maclaurin series generated by $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{n!\, a_n}{n!} x^n = \sum_{n=0}^{\infty} a_n x^n$

58 $\quad f(x)=\sum_{n=0}^{\infty} a_n(x-a)^n$ on $|x-a|<h$

$\Rightarrow f^{(k)}(x)=\sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1)\,a_n(x-a)^{n-k}$ on $|x-a|<h$

$\Rightarrow f^{(k)}(a)=k!\,a_k$

$\Rightarrow$ Taylor series generated by f at $a=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n=\sum_{n=0}^{\infty}a_n(x-a)^n$

s9.1-extra 1

$$f(x)=\begin{cases} 0 &, \text{ if } x=0 \\ e^{-\frac{1}{x^2}} &, \text{ if } x\neq 0 \end{cases}$$

(i) $$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{e^{-\frac{1}{h^2}}}{h}=\lim_{h\to 0}\frac{\frac{1}{h}}{e^{\frac{1}{h^2}}}$$

$$=\lim_{h\to 0}\frac{\frac{-1}{h^2}}{e^{\frac{1}{h^2}}\cdot\frac{-2}{h^3}}=\lim_{h\to 0}\frac{h}{2e^{\frac{1}{h^2}}}=0$$

and $f'(x)=e^{-\frac{1}{x^2}}\cdot\frac{2}{x^3}$ when $x\neq 0$

(ii) $$f''(0)=\lim_{h\to 0}\frac{f'(0+h)-f'(0)}{h}=\lim_{h\to 0}\frac{\frac{2}{h^3}e^{-\frac{1}{h^2}}}{h}=\lim_{h\to 0}\frac{\frac{2}{h^4}}{e^{\frac{1}{h^2}}}=\lim_{h\to 0}\frac{\frac{-8}{h^5}}{e^{\frac{1}{h^2}}\cdot\frac{-2}{h^3}}$$

$$=\lim_{h\to 0}\frac{\frac{4}{h^2}}{e^{\frac{1}{h^2}}}=\lim_{h\to 0}\frac{\frac{-8}{h^3}}{e^{\frac{1}{h^2}}\cdot\frac{-2}{h^3}}=\lim_{h\to 0}\frac{4}{e^{\frac{1}{h^2}}}=0$$

§9.1- extra2

(a) $\frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt$

$\left(\begin{array}{l} \text{let } u=(x-t)^n,\ dv=f^{(n+1)}(t)\,dt \\ \Rightarrow du=-n(x-t)^{n-1},\ v=f^{(n)}(t) \end{array}\right)$

$$= \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{1}{n!}\left((x-t)^n f^{(n)}(t)\Big|_a^x + \int_a^x n(x-t)^{n-1} f^{(n)}(t)\,dt\right)$$

$$= \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{1}{n!}\left(-(x-a)^n f^{(n)}(a) + n\int_a^x f^{(n)}(t)(x-t)^{n-1}\,dt\right) = \frac{1}{(n-1)!}\int_a^x f^{(n)}(t)(x-t)^{n-1}\,dt$$

$$\Rightarrow \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x) = R_{n-1}(x)$$

$$\Rightarrow f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+R_n(x)$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + R_{n-1}(x)$$

$$= \cdots = f(a) + R_0(x) = f(a) + \int_a^x f'(t)\,dt = f(x)$$

(b) $\because$ $(x-t)^n$ doesn't change sign for t between a and x

$$\therefore \int_a^x f^{(n+1)}(t)(x-t)^n\,dt = f^{(n+1)}(c)\int_a^x (x-t)^n\,dt \quad \text{for some } c \text{ between } a \text{ and } x$$

$$= f^{(n+1)}(c)\left[\frac{-1}{n+1}(x-t)^{n+1}\right]_a^x$$

$$= \frac{f^{(n+1)}(c)}{n+1}(x-a)^{n+1}$$

$$\Rightarrow R_n(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt = \frac{1}{n!}\frac{f^{(n+1)}(c)}{n+1}(x-a)^{n+1}$$

$$= \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x$$

§9.8-extra1 (a) $(k+1)\binom{m}{k+1}+k\binom{m}{k}=(k+1)\dfrac{m!}{(k+1)!(m-k-1)!}+k\binom{m}{k}$

$$=\frac{m!}{k!(m-k-1)!}\cdot\frac{m-k}{m-k}+k\binom{m}{k}$$

$$=(m-k)\binom{m}{k}+k\binom{m}{k}=m\binom{m}{k}$$

(b) $f(x)=\sum_{k=0}^{\infty}\binom{m}{k}x^k$ on $|x|<1$

$\Rightarrow f'(x)=\sum_{k=1}^{\infty}k\binom{m}{k}x^{k-1}$ on $|x|<1$ and $f(0)=\binom{m}{0}=1$

$\Rightarrow (1+x)f'(x)=\sum_{k=1}^{\infty}k\binom{m}{k}x^{k-1}+\sum_{k=1}^{\infty}k\binom{m}{k}x^k=\sum_{k=0}^{\infty}(k+1)\binom{m}{k+1}x^k+\sum_{k=1}^{\infty}k\binom{m}{k}x^k$

$$=m+\sum_{k=1}^{\infty}\left[(k+1)\binom{m}{k+1}+k\binom{m}{k}\right]x^k=m+\sum_{k=1}^{\infty}m\binom{m}{k}x^k$$

$$=m\sum_{k=0}^{\infty}\binom{m}{k}x^k=mf(x)$$

(c) $(1+x)\frac{df}{dx} = mf \Rightarrow \int \frac{1}{f}\,df = \int \frac{m}{1+x}\,dx$

$\Rightarrow \ln|f| = m\ln|1+x| + C$

$\Rightarrow |f| = |1+x|^m \cdot e^c$ on $|x|<1$

$\Rightarrow f(x) = (1+x)^m \cdot \tilde{c}$ on $|x|<1$

$f(0)=1 \Rightarrow \tilde{c}=1 \Rightarrow f(x) = (1+x)^m$ on $|x|<1$