58.6-extra1

(i) For $\int_0^1 x^{-p}\,dx$

① $p=1$, $\int_0^1 x^{-1}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{x}\,dx = \lim_{a\to 0^+}\left(\ln x\Big|_a^1\right) = \lim_{a\to 0^+}(-\ln a) = \infty$

$\Rightarrow$ The improper integral $\int_0^1 x^{-1}\,dx$ does not exist.

② $p\neq 1$, $\int_0^1 x^{-p}\,dx = \lim_{a\to 0^+}\int_a^1 x^{-p}\,dx = \lim_{a\to 0^+}\left(\frac{1}{1-p}x^{1-p}\Big|_a^1\right) = \lim_{a\to 0^+}\left(\frac{1}{1-p} - \frac{a^{1-p}}{1-p}\right)$

$= \begin{cases} \infty, & p>1 \\ \frac{1}{1-p}, & 0<p<1 \end{cases}$ $p>1 \Rightarrow$ Improper integral does not exist

(ii) For $\int_1^\infty x^{-p}\,dx$

① $p=1$, $\int_1^\infty x^{-1}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\left(\ln x\Big|_1^b\right) = \lim_{b\to\infty}(\ln b) = \infty$

$\Rightarrow$ The improper integral does not exist

② $p\neq 1$, $\int_1^\infty x^{-p}\,dx = \lim_{b\to\infty}\int_1^b x^{-p}\,dx = \lim_{b\to\infty}\left(\frac{1}{1-p}x^{1-p}\Big|_1^b\right) = \lim_{b\to\infty}\left(\frac{b^{1-p}}{1-p} - \frac{1}{1-p}\right)$

$$= \begin{cases} \frac{1}{p-1}, & p>1 \\ \infty, & 0<p<1 \Rightarrow \text{Improper integral does not exist.} \end{cases}$$

8.6-38

$$\int_0^1 \frac{1}{t-\sin t}\,dt$$

Since $t \geq \sin t \geq 0$ for $0 \leq t \leq 1 \Rightarrow 0 \leq t-\sin t \leq t$

$$\Rightarrow 0 \leq \frac{1}{t} \leq \frac{1}{t-\sin t}$$

And $\int_0^1 \frac{1}{t}\,dt = \lim_{b\to 0^+}[\ln|t|]_b^1 = \lim_{b\to 0^+}[-\ln b] = \infty$ diverges

$\Rightarrow \int_0^1 \frac{1}{t-\sin t}\,dt$ diverges

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$$\int_2^\infty \frac{1}{\ln x}\,dx$$

Since $0 < \frac{1}{x} < \frac{1}{\ln x}$ for $x \geq 2$, and $\int_2^\infty \frac{1}{x}\,dx$ diverges

$\Rightarrow \int_2^\infty \frac{1}{\ln x}\,dx$ diverges

56 $\int_{e^e}^{\infty} \ln(\ln x)\,dx$

since $\ln(\ln x) \geq 1$ for $x \geq e^e$, and $\int_{e^e}^{\infty} 1\,dx$ diverges

$\Rightarrow \int_{e^e}^{\infty} \ln(\ln x)\,dx$ diverges

58 $\int_1^{\infty} \frac{1}{e^x - 2^x}\,dx$

Since $\lim_{x\to\infty} \frac{\frac{1}{e^x-2^x}}{\frac{1}{e^x}} = \lim_{x\to\infty} \frac{e^x}{e^x-2^x} = \lim_{x\to\infty} \frac{1}{1-(\frac{2}{e})^x} = 1$

and $\int_1^{\infty} \frac{1}{e^x}\,dx = \lim_{b\to\infty} \left[-\frac{1}{e^x}\right]_1^b = \lim_{b\to\infty} \left(-\frac{1}{e^b} + \frac{1}{e}\right) = \frac{1}{e}$ converges

$\Rightarrow \int_1^{\infty} \frac{1}{e^x-2^x}\,dx$ converges

60 $\int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}}\,dx = 2\int_0^{\infty} \frac{1}{e^x+e^{-x}}\,dx$

Since $0 \leq \frac{1}{e^x+e^{-x}} < \frac{1}{e^x}$ for $x>0$, and $\int_0^{\infty} \frac{1}{e^x}\,dx$ converges

$\Rightarrow \int_0^{\infty} \frac{1}{e^x+e^{-x}}\,dx$ converges $\Rightarrow \int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}}\,dx$ converges

71. $\int_0^{\infty} \frac{2x}{x^2+1}\,dx = \int_1^{\infty} \frac{1}{y}\,dy$ (Let $y = x^2+1$)

$= \lim_{b\to\infty} \ln|y|\Big|_1^b = \lim_{b\to\infty} \ln b = \infty$ diverges

But $\lim_{b\to\infty} \int_{-b}^{b} \frac{2x}{x^2+1}\,dx = \lim_{b\to\infty} \int_{b^2+1}^{b^2+1} \frac{1}{y}\,dy = 0$

Hence $\int_{-\infty}^{\infty} f(x)\,dx$ may not equal $\lim_{b\to\infty} \int_{-b}^{b} f(x)\,dx$

73. $\int_{-1}^{3} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{3} \frac{1}{x^2}\,dx$

And $\int_{-1}^{0} \frac{1}{x^2}\,dx = \lim_{b \to 0^-} \left[-\frac{1}{x}\right]_{-1}^{b} = \lim_{b \to 0^-} \left[-\frac{1}{b} - 1\right] = \infty$ diverges

$\Rightarrow \int_{-1}^{3} \frac{1}{x^2}\,dx$ diverges

The calculation in text book is wrong!

§8.6-extra2

$$\because \lim_{x\to\frac{\pi}{2}} \frac{e^x \tan^p x}{\frac{1}{(\frac{\pi}{2}-x)^p}} = \lim_{x\to\frac{\pi}{2}} e^x \sin^p x \left(\frac{\frac{\pi}{2}-x}{\cos x}\right)^p = e^{\frac{\pi}{2}}\cdot 1\cdot 1 = e^{\frac{\pi}{2}}$$

where $\displaystyle\lim_{x\to\frac{\pi}{2}} \frac{\frac{\pi}{2}-x}{\cos x} = \lim_{x\to\frac{\pi}{2}} \frac{-1}{-\sin x} = 1$

$\therefore$ $\displaystyle\int_0^{\frac{\pi}{2}} e^x \tan^p x\, dx$ and $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{(\frac{\pi}{2}-x)^p}\, dx$ both converge or diverge

① $p=1$, $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{\frac{\pi}{2}-x}\, dx = \lim_{a\to\frac{\pi}{2}^-} -\ln\left|\frac{\pi}{2}-x\right| \Big|_0^a = \lim_{a\to\frac{\pi}{2}^-} \left(\ln\frac{\pi}{2} - \ln\left|\frac{\pi}{2}-a\right|\right) = \infty$

$\Rightarrow \displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{\frac{\pi}{2}-x}\, dx$ is divergent

$\Rightarrow \displaystyle\int_0^{\frac{\pi}{2}} e^x \tan x\, dx$ is divergent

② $p\neq 1$, $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{(\frac{\pi}{2}-x)^p}\, dx = \lim_{a\to\frac{\pi}{2}^-} \frac{-1}{-p+1}\left(\frac{\pi}{2}-x\right)^{-p+1}\Big|_0^a = \lim_{a\to\frac{\pi}{2}^-} \frac{1}{p-1}\left(\left(\frac{\pi}{2}-a\right)^{-p+1} - \left(\frac{\pi}{2}\right)^{-p+1}\right)$

$$= \begin{cases} \infty, & p>1 \\ \frac{1}{1-p}\left(\frac{\pi}{2}\right)^{1-p}, & 0<p<1 \end{cases}$$

By ①,② $\Rightarrow$ $\displaystyle\int_0^{\frac{\pi}{2}} e^x \tan^p x\, dx$ is convergent if $0<p<1$
and is divergent if $p\geq 1$

9.1-60 $\because \lim_{n\to\infty} \frac{x^n}{n!} = 0$, $\forall x \in \mathbb{R}$

$\therefore \lim_{n\to\infty} \frac{n!}{2^n \cdot 3^n} = \lim_{n\to\infty} \frac{1}{\frac{6^n}{n!}} = \infty \Rightarrow \{a_n\}$ diverges

71. (a) True, by definition

(b) False, Ex: $a_n = (-1)^n$

(c) False, Ex: $a_n = (-1)^n$ diverges

but $a_{2n} = (-1)^{2n} = 1$ is a convergent subsequence of a_n

72. (a) True, if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$

then $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = A+B$ converges

(b) False, Ex: $a_n = n$, $b_n = -n$ both diverge

but $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} 0 = 0$ converges

(c) False, Ex: $a_n = n$ diverges and $b_n = 0$

then $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} 0 = 0$ converges

9.2-14

$\sum_{n=0}^{\infty}\left(\frac{2^{n+1}}{5^n}\right) = \sum_{n=0}^{\infty} 2 \cdot \left(\frac{2}{5}\right)^n = \frac{2}{1-\frac{2}{5}} = \frac{10}{3}$

30 $\because \lim_{n\to\infty} \ln\frac{1}{n} = -\infty \neq 0$, $\therefore \sum_{n=1}^{\infty} \ln\frac{1}{n}$ diverges

62 It's possible. Ex: $a_n = 1$, $b_n = -1$, $\forall n \in \mathbb{N}$

$\Rightarrow \Sigma a_n$ and Σb_n both diverge, but $\Sigma(a_n + b_n) = 0$ converges

63. If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0 \Rightarrow \lim_{n\to\infty} \frac{1}{a_n} = \infty$

$\Rightarrow \sum \frac{1}{a_n}$ diverges

9.3-24 $\because \lim_{n\to\infty} \frac{\frac{1}{(\ln n)^2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{(\ln n)^2} = \lim_{n\to\infty} \frac{1}{2\ln n \cdot \frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2\ln n} = \lim_{n\to\infty} \frac{1}{2\cdot\frac{1}{n}} = \infty$

and $\sum \frac{1}{n}$ diverges $\Rightarrow \sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$ diverges

62 (a) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)} \Rightarrow p=1$

By integral test and 61 (a) $\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$ diverges

(b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1.01}} \Rightarrow p=1.01>1$

By integral test and 61 (a) $\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1.01}}$ converges

(c) $\sum_{n=2}^{\infty} \frac{1}{n \ln n^3} = \sum_{n=2}^{\infty} \frac{1}{3n \ln n} \Rightarrow p=1$

By integral test and 61 (a) $\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n \ln n^3}$ diverges

(d) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3} \Rightarrow p=3$

By integral test and 61 (a) $\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ converges

§9.3-extra 1 (i) $\because \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

$\therefore \sum_{n=1}^{\infty} \sin\frac{1}{n}$ is divergent

(ii) $\because \lim_{n\to\infty} \frac{1-\cos\frac{1}{n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{-1}{n^2}\sin\frac{1}{n}}{\frac{-2}{n^3}} = \lim_{n\to\infty} \frac{1}{2}\cdot\frac{\sin\frac{1}{n}}{\frac{1}{n}} = \frac{1}{2}$