Final

1. $D=\{(x,y,z) \mid 0\le x\le 1,\ 0\le xy\le 1,\ 0\le z\le 1\}$

Let $u=xy \Rightarrow y=\frac{u}{x}$

$$\Rightarrow J(x,u,z)=\begin{vmatrix} 1 & 0 & 0 \\ \frac{-u}{x^2} & \frac{1}{x} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{x}$$

$$\Rightarrow \iiint_D (x^2y+xyz)\,dV = \int_0^1\int_0^1\int_0^1 (xu+uz)\left|\frac{1}{x}\right| dz\,du\,dx$$

$$=\int_0^1\int_0^1 u+\frac{u}{2x}\,du\,dx$$

$$=\int_0^1 \frac{1}{2}+\frac{1}{4}\frac{1}{x}\,dx = \frac{1}{2}x+\frac{1}{4}\ln x\Big|_0^1$$

$$=\frac{1}{2}-\lim_{x\to 0^+}\frac{1}{4}\ln x=\infty$$

2. $\vec{r}(t)=(2-t,\ 2-t,\ 2-t),\quad 0\le t\le 1$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_0^1 ((2-t)^2, (2-t)^2, (2-t)^2)\cdot(-1,-1,-1)\,dt$$

$$= -3\int_0^1 4-4t+t^2\,dt = -7$$

3. Let $f(x,y,z)=x+y+z \Rightarrow \nabla f=(1,1,1)$

project to x-y plane $\Rightarrow \vec{p}=(0,0,1) \Rightarrow \frac{|\nabla f|}{|\nabla f\cdot \vec{p}|}=\sqrt{3}$

$$\Rightarrow \iint_{x+y+z=1} 2x-y+z\,d\sigma = \int_0^1\int_0^{1-y}[2x-y+(1-x-y)]\sqrt{3}\,dx\,dy = \int_0^1\int_0^{1-y}\sqrt{3}(1+x-2y)\,dx\,dy$$

$$=\sqrt{3}\int_0^1 (1-y)+\frac{1}{2}(1-y)^2-2y(1-y)\,dy = \sqrt{3}\int_0^1 \frac{5}{2}y^2-4y+\frac{3}{2}\,dy$$

$$=\frac{\sqrt{3}}{3}$$

4. (a) $F(x,y) = (2y, x)$

$\because \frac{\partial(2y)}{\partial y} = 2$ but $\frac{\partial x}{\partial x} = 1 \neq 2$, $\therefore F$ is not conservative on $\mathbb{R}$

(b) $G(x,y) = (x,y)$

$\because \frac{\partial x}{\partial y} = 0 = \frac{\partial y}{\partial x} \Rightarrow$ possible to find $g(x,y)$ s.t. $\nabla g = G$

$g_x = x \Rightarrow g(x,y) = \frac{x^2}{2} + C_1(y) \Rightarrow g_y = C_1'(y) = y$

$\Rightarrow C_1(y) = \frac{y^2}{2} + C \Rightarrow g(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + C$ for some $C \in \mathbb{R}$

$\because \exists\, g(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + C$ s.t. $G = \nabla g$ on $\mathbb{R}$

$\therefore G$ is conservative on $\mathbb{R}$

(c) $H(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$

$\because \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$

$\Rightarrow$ possible to find $h(x,y)$ s.t. $\nabla h = H$

$h_x = \frac{-y}{x^2+y^2} \Rightarrow h(x,y) = -\tan^{-1}\frac{x}{y} + C_2(y) \Rightarrow h_y = \frac{x}{x^2+y^2} + C_2'(y) = \frac{x}{x^2+y^2}$

$\Rightarrow C_2(y) = C \Rightarrow h(x,y) = -\tan^{-1}\left(\frac{x}{y}\right) + C$ for some $C \in \mathbb{R}$

but h is not defined at $y=0 \Rightarrow \nabla h \neq H$ on $\mathbb{R}$

$\Rightarrow H$ is not conservative on $\mathbb{R}$

5. (a) We can apply Stokes Theorem on S.

(i) let $\vec{r}(t) = (\cos t, \sin t, 0)$, $0 \le t \le 2\pi$

$$\Rightarrow \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (\cos t, 0, \cos t \sin t) \cdot (-\sin t, \cos t, 0)\, dt$$

$$= \int_0^{2\pi} -\sin t \cos t\, dt = 0$$

(ii) let $\vec{r}(\theta, \phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$, $0 \le \phi \le \frac{\pi}{2}$, $0 \le \theta \le 2\pi$

$$\Rightarrow |\vec{r}_\theta \times \vec{r}_\phi| = |\sin\phi|$$

$$\Rightarrow \iint_S \nabla\times\vec{F} \cdot \vec{n}\, d\sigma = \iint_S (0, 1-y, z) \cdot \frac{(x, y, z)}{\sqrt{x^2+y^2+z^2}}\, d\sigma$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} [(1 - \sin\phi\sin\theta)\sin\phi\sin\theta + \cos^2\phi] \sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin^2\phi\sin\theta - \sin^3\phi\sin^2\theta + \cos^2\phi\sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{1-\cos 2\phi}{2}\sin\theta - (1-\cos^2\phi)\sin\phi\sin^2\theta + \cos^2\phi\sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{\phi}{2}\sin\theta - \frac{1}{4}\sin 2\phi\sin\theta + \cos\phi\sin^2\theta - \frac{1}{3}\cos^3\phi\sin^2\theta - \frac{1}{3}\cos^3\phi\right]_0^{\frac{\pi}{2}} d\theta$$

$$= \int_0^{2\pi} \frac{\pi}{4}\sin\theta - \sin^2\theta + \frac{1}{3}\sin^2\theta + \frac{1}{3}\, d\theta = \int_0^{2\pi} -\frac{2}{3}\,\frac{1-\cos 2\theta}{2} + \frac{1}{3}\, d\theta = 0$$

(b) We can apply divergence Theorem on D

(i) $\iiint_D \nabla\cdot\vec{F}\, dV = \iiint_D 1\, dV = \frac{4}{3}\pi 2^3 \cdot \frac{1}{2} = \frac{16}{3}\pi$

(ii) Let $S_1 = \{(x,y,z) \mid x=0,\ y^2+z^2 \le 4\}$
$S_2 = \{(x,y,z) \mid x^2+y^2+z^2 = 4,\ x > 0\}$

$$\Rightarrow \iint_{\partial D} \vec{F}\cdot\vec{n}\,d\sigma = \iint_{S_1} \vec{F}\cdot\vec{n}\,d\sigma + \iint_{S_2} \vec{F}\cdot\vec{n}\,d\sigma$$

① $\vec{r}_1(r,\theta) = (0, r\cos\theta, r\sin\theta) \Rightarrow |(\vec{r}_1)_r \times (\vec{r}_1)_\theta| = |r|$

$$\Rightarrow \iint_{S_1} \vec{F}\cdot\vec{n}\,d\sigma = \iint_{S_1} -x-z\,d\sigma = \int_0^{2\pi}\int_0^2 -r\sin\theta\; r\,dr\,d\theta$$

$$= \int_0^{2\pi} -\frac{8}{3}\sin\theta\,d\theta = 0$$

② $\vec{r}_2(\phi,\theta) = (2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi)$, $0\le\phi\le\pi$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

$$\Rightarrow |(\vec{r}_2)_\phi \times (\vec{r}_2)_\theta| = |4\sin\phi|$$

$$\Rightarrow \iint_{S_2} \vec{F}\cdot\vec{n}\,d\sigma = \iint_{S_2} (x+z, xz, xy)\cdot\frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}\,d\sigma = \frac{1}{2}\iint_{S_2} x^2+xz+2xyz\,d\sigma$$

$$= \frac{1}{2}\int_0^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [4\sin^2\phi\cos^2\theta + 4\sin\phi\cos\phi\cos\theta + 16\sin^2\phi\cos\phi\sin\theta\cos\theta]\,4\sin\phi\,d\theta\,d\phi$$

$$= \frac{1}{2}\int_0^{\pi} [8\sin^3\phi\,\theta + 4\sin^3\phi\sin 2\theta + 16\sin^2\phi\cos\phi\sin\theta + 8\sin^3\phi\cos\phi\sin^2\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\,d\phi$$

$$= \frac{1}{2}\int_0^{\pi} 8\pi\sin\phi(1-\cos^2\phi) + 32\sin^2\phi\cos\phi\,d\phi$$

$$= \frac{1}{2}\left[-8\pi\cos\phi + \frac{8}{3}\pi\cos^3\phi + \frac{32}{3}\sin^3\phi\right]_0^{\pi} = \frac{16}{3}\pi$$

6. $\vec{F} = \dfrac{(x,y,z)}{(\sqrt{x^2+y^2+z^2})^3} \Rightarrow \nabla\cdot\vec{F} = 0$

$$\Rightarrow \iint_{\frac{(x-1)^2}{4}+y^2+z^2=1} \vec{F}\cdot\vec{n}\,d\sigma = \iiint_{\frac{(x-1)^2}{4}+y^2+z^2\le 1} \nabla\cdot\vec{F}\,dV = 0$$