1.

Example : $f(x) = x + \frac{1}{x}$

Vertical asymptote : $x=0$ $\left(\lim_{x\to 0^+} f(x) = \infty \right.$

$\left. \lim_{x\to 0^-} f(x) = -\infty \right)$

oblique asymptote : ~~f(x) = x~~ $y=x$

$\left(\lim_{x\to\infty} f(x) - x = 0 \right.$

$\left. \lim_{x\to\infty^-} f(x) - x = 0 \right)$

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2. Distance from (x,y) to $(0,\frac{3}{2})$ is equal to $\sqrt{(x-0)^2+(y-\frac{3}{2})^2}$

$y=x^2 \Rightarrow$ It's equal to find $\min\left(\sqrt{(x-0)^2+(x^2-\frac{3}{2})^2}\right)$

$\left(\because (x-0)^2+(x^2-\frac{3}{2})^2\right.$ 恆正 $\Rightarrow$ 求 $\min\left((x-0)^2+(x^2-\frac{3}{2})^2\right)$ 之 x 即可 $)$

$\Rightarrow$ 令 $f(x) = (x-0)^2+(x^2-\frac{3}{2})^2 = x^4-2x^2+\frac{9}{4}$

$f'(x) = 4x^3-4x = 4x(x+1)(x-1)$, extreme : $x=0$ or ± 1

代入即知 minimum 發生在 $x=\pm 1 \Rightarrow$ 所求 $= \sqrt{(1)+(\frac{1}{4})} = \frac{\sqrt{5}}{2}$ #

3. Rolle's Theorem

If $f(x)$ is continuous in $[a,b]$, and differentiable in (a,b).

If $f(a)=f(b)=0$, then there is at least one c between a and b.

such that $f'(c)=0$

Mean Value Theorem

If $f(x)$ is continuous in $[a,b]$, and differentiable in (a,b).

Then there exists one c between a and b such that

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

Rolle's Theorem $\Rightarrow$ Mean Value Theorem

Pf: Let $h(x)=f(x)-\left[f(a)+\frac{f(b)-f(a)}{b-a}(x-a)\right]$

$\Rightarrow h$ is continuous in $[a,b]$ and differentiable in (a,b).

Moreover, $h(a)=h(b)=0$.

By Rolle's Theorem, $\exists\, c\in(a,b)$ s.t. $h'(c)=0$

$$\Rightarrow f'(c)=\frac{f(b)-f(a)}{b-a}$$

Mean Value Theorem $\Rightarrow$ Rolle's Theorem

Pf: Since $f(x)$ is continuous in $[a,b]$, and differentiable in (a,b)

and $f(a)=f(b)=0$

By Mean Value Theorem, $\exists\ c\in(a,b)$ s.t.

$$f'(c)=\frac{f(b)-f(a)}{b-a}=0$$

Hence these two theorems are equivalent to each other.

4. (i) Let $P = \{x_0, x_1, \cdots, x_n\}$ be a partition on $[0,1]$

and $\Delta x_k = x_k - x_{k-1}$, $c_k \in [x_{k-1}, x_k]$

If $\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$ exists, then

$$\int_0^1 f(x)\,dx = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

(ii)

$P = \{0, 0.1, 0.4, 0.6, 0.8, 1\}$

5. $$\lim_{n \to \infty} \sum_{k=1}^{2n} \frac{1}{2n}\sqrt{1 - \left(\frac{k-\frac{1}{2}}{2n}\right)^2} = \int_0^1 \sqrt{1-x^2}\,dx = \pi \cdot 1^2 \cdot \frac{1}{4} = \frac{\pi}{4}$$