1. $\lim\limits_{x\to\infty}\left(\sqrt{x^2+x}-x\right)=\lim\limits_{x\to\infty}\dfrac{\left(\sqrt{x^2+x}-x\right)\left(\sqrt{x^2+x}+x\right)}{\sqrt{x^2+x}+x}=\lim\limits_{x\to\infty}\dfrac{x}{\sqrt{x^2+x}+x}$

$=\lim\limits_{x\to\infty}\dfrac{1}{\sqrt{1+\frac{1}{x}}+1}=\dfrac{1}{2}$

$\lim\limits_{\theta\to 0}\dfrac{\sin(\tan 2\theta)}{\theta}=\lim\limits_{\theta\to 0}\dfrac{\sin(\tan 2\theta)}{\tan 2\theta}\cdot\dfrac{\tan 2\theta}{\theta}=\lim\limits_{\theta\to 0}\dfrac{\sin(\tan 2\theta)}{\tan 2\theta}\,\dfrac{\sin 2\theta}{2\theta}\cdot\dfrac{2}{\cos(2\theta)}$

$=2$

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(a) If $f(x)$ is a continuous function on $[a, b]$,

then $\forall M \in [f(a), f(b)]$, $\exists c \in [a, b]$ such that $f(c) = M$

(b)

Let $f(x) = x - 1 - \cos x$

Since $f(1) = 1 - 1 - \cos 1 = -\cos 1 < 0$

& $f(2) = 2 - 1 - \cos 2 = 1 - \cos 2 > 0$

By the Intermediate Value Theorem, we can find $t \in [1, 2]$

such that $f(t) = 0$

3. (a) Given $\epsilon > 0$, $\exists\, \delta > 0$ such that for all x

$$0 < |x-1| < \delta \Rightarrow \left|\frac{1}{x} - 1\right| < \epsilon$$

(b)

$$\left|\frac{1}{x} - 1\right| < 0.1 \iff -0.1 < \frac{1}{x} - 1 < 0.1$$

$$\iff 0.9 < \frac{1}{x} < 1.1$$

$$\iff \frac{1}{1.1} < x < \frac{1}{0.9} \iff \underbrace{-\frac{0.1}{1.1}}_{= -\frac{1}{11}} < x - 1 < \underbrace{\frac{0.1}{0.9}}_{= \frac{1}{9}}$$

Hence we take $\delta = \frac{1}{11}$

4.

(a) $\lim_{x \to 0^-} f(x) = 1$

$\Rightarrow$ Given $\epsilon > 0$, $\exists\, \delta > 0$ such that for all x

$-\delta < x - 0 < 0 \Rightarrow |f(x) - 1| < \epsilon$

(b) $\lim_{x \to \infty} f(x) = 1$

$\Rightarrow$ Given $\epsilon > 0$, $\exists\, M \in \mathbb{R}$ such that for all x

$x > M \Rightarrow |f(x) - 1| < \epsilon$

5. $\because$ f, g are continuous at $x=c$

$\therefore$ $\forall\, \epsilon>0$, $\exists\, \delta_1>0$ s.t. if $|x-c|<\delta_1 \Rightarrow |f(x)-f(c)|<\frac{\epsilon}{2}$

$\exists\, \delta_2>0$ s.t. if $|x-c|<\delta_2 \Rightarrow |g(x)-g(c)|<\frac{\epsilon}{4}$

pick $\delta=\min\{\delta_1,\delta_2\}>0$

then if $|x-c|<\delta \Rightarrow |(f+2g)(x)-(f+2g)(c)|$

$\leq |f(x)-f(c)|+2|g(x)-g(c)|<\frac{\epsilon}{2}+2\cdot\frac{\epsilon}{4}=\epsilon$

Hence $f+2g$ is continuous at $x=c$