Midterm II

1. $y = \frac{x^3+2x-2}{x+1} = x^2-x+3-\frac{5}{x+1} \Rightarrow \frac{dy}{dx} = 2x-1+\frac{5}{(x+1)^2}$

$\Rightarrow \frac{d^2y}{dx^2} = 2 - \frac{10}{(x+1)^3}$

$\lim_{x\to\infty}\left(x^2-x+3-\frac{5}{x+1}\right)=\infty$ and $\lim_{x\to-\infty}\left(x^2-x+3-\frac{5}{x+1}\right)=\infty$

$\Rightarrow$ when $x\to\infty$, x^2-x+3 dominates

when $x\to-\infty$, x^2-x+3 dominates

$\lim_{x\to-1^+}\left(x^2-x+3-\frac{5}{x+1}\right)=-\infty$, $\lim_{x\to-1^-}\left(x^2-x+3-\frac{5}{x+1}\right)=\infty$

$\Rightarrow$ When x close to -1, $-\frac{5}{x+1}$ dominates

and $x=-1$ is a vertical asymptote

$\frac{dy}{dx}=0 \Rightarrow 2x-1+\frac{5}{(x+1)^2}=0$ and $x\neq-1$

$\Rightarrow 2x^3+3x^2+4=0 \Rightarrow (x+2)(2x^2-x+2)=0$

$\because 2x^2-x+2 = 2\left(\left(x-\frac{1}{4}\right)^2+\frac{15}{16}\right)>0 \Rightarrow \frac{dy}{dx}=0$ implies $x=-2$

$\Rightarrow$ Critical points : $x=-1$ and $x=-2$

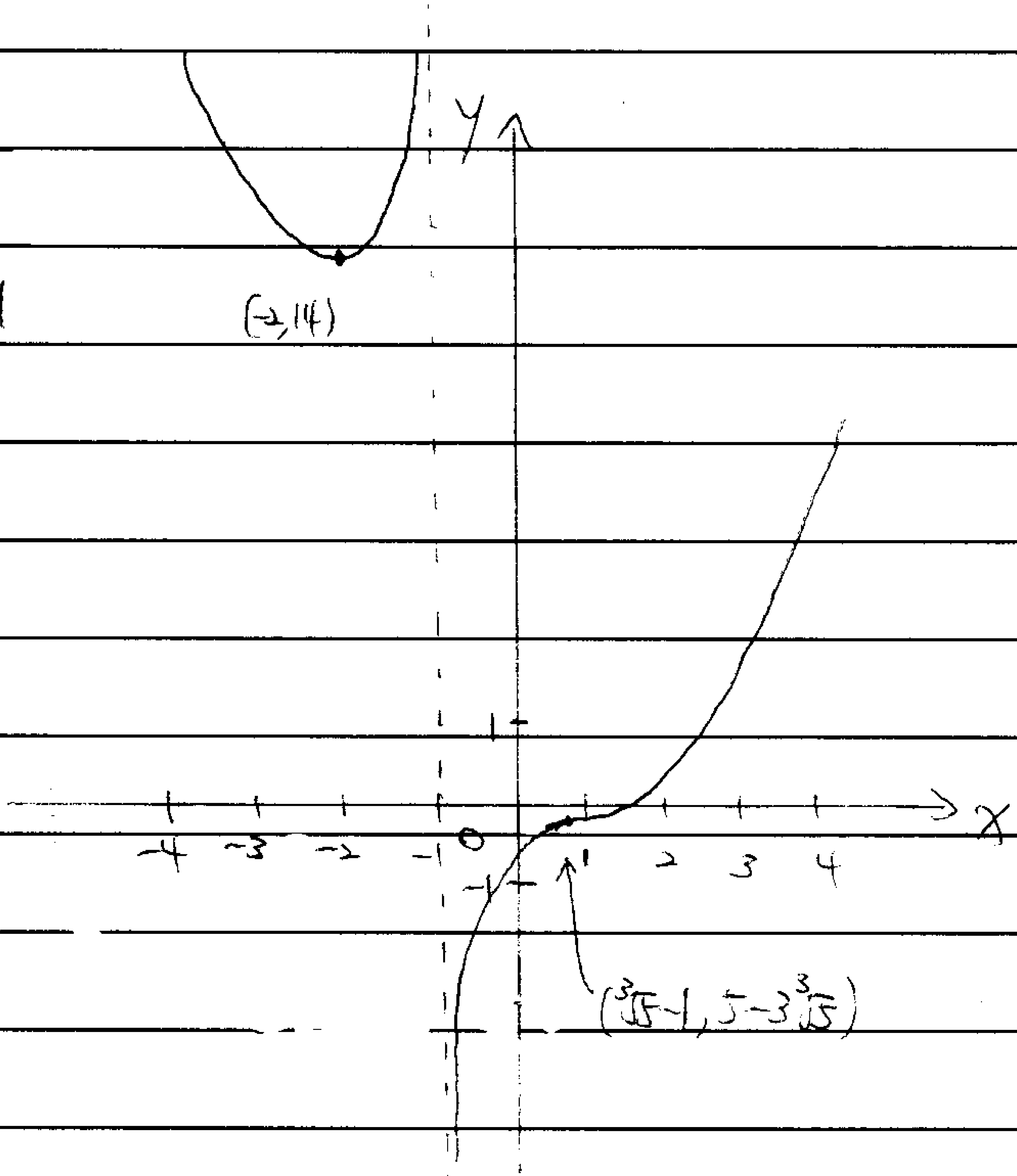
$\frac{d^2y}{dx^2}=0 \Rightarrow 2-\frac{10}{(x+1)^3}=0$ and $x\neq-1 \Rightarrow (x+1)^3=5 \Rightarrow x=\sqrt[3]{5}-1$

$\Rightarrow$ Inflection points : $x=\sqrt[3]{5}-1$

$\Rightarrow y(-2)=14$ and $y(\sqrt[3]{5}-1) = 5-3\sqrt[3]{5}<0$

Per-Duet

$\frac{dy}{dx}$	$-$	$+$	$+$	$+$
$\frac{d^2y}{dx^2}$	$+$	$+$	$-$	$+$
	$x<-2$	$-2<x<-1$	$-1<x<\sqrt[3]{5}-1$	$x>\sqrt[3]{5}-1$



2. $d^2 = (x-1)^2 + (y-\sqrt{3})^2 = (x^2-2x+1) + (y^2-2\sqrt{3}y+3)$

$(y=\sqrt{16-x^2}) \quad = x^2-2x+1+16-x^2-2\sqrt{3}\sqrt{16-x^2}+3$

$= 20 - 2x - 2\sqrt{3}\sqrt{16-x^2} \overset{\text{let}}{=\!=} f(x)$

$f'(x) = -2-2\sqrt{3}\left(\frac{-x}{\sqrt{16-x^2}}\right) = -2 + \frac{2\sqrt{3}x}{\sqrt{16-x^2}}$

$f'(x)=0 \Rightarrow 2\sqrt{3}x = 2\sqrt{16-x^2} \Rightarrow 12x^2 = 64-4x^2 \Rightarrow x = \pm 2$ (critical)

Endpoint : $f(4) = 12$, $f(-4) = 28$

Critical point : $f(2) = 4$, $f(-2) = 16$

We have minimal distance $\sqrt{4} = 2$

3. (a) let $y = x^x \Rightarrow \ln y = \ln x^x = x \ln x$

$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} -x = 0$

$\Rightarrow \lim_{x\to 0^+} x^x = e^0 = 1$

(b) $\lim_{x\to 0} \frac{x^2\cos\frac{1}{x}}{\sin x} = \lim_{x\to 0} \frac{x}{\sin x} \cdot x \cdot \cos\frac{1}{x} = 0$

4. (a) $\because f'(x) = e^{-\frac{x^2}{2}} > 0$, $\forall x \in \mathbb{R}$

$\Rightarrow f$ is strictly increasing on $\mathbb{R}$

$\Rightarrow f$ is one-to-one

(b) $f(0) = 1 + \int_0^0 e^{-\frac{t^2}{2}} dt = 1$

$f'(x) = e^{-\frac{x^2}{2}} \Rightarrow f'(0) = e^0 = 1$

(c) $\because f(0) = 1 \Rightarrow \frac{d}{dy} f^{-1}(y)\Big|_{y=1} = \frac{1}{\frac{df(x)}{dx}\Big|_{x=0}} = \frac{1}{f'(0)} = 1$

5. Let $u = \sin x \Rightarrow \frac{du}{dx} = \cos x$

$\frac{d}{dx} \int_{\sin x}^{1} e^{t^2} dx = -\frac{d}{dx} \int_1^{\sin x} e^{t^2} dt = -\frac{d \int_1^u e^{t^2} dt}{du} \cdot \frac{du}{dx}$

$= -e^{u^2} \cdot \cos x = -\cos x \, e^{\sin^2 x}$

6. (a) $\int_1^2 \frac{1}{x(1+(\ln x)^2)} dx$, let $u = \ln x \Rightarrow du = \frac{1}{x} dx$

$= \int_0^{\ln 2} \frac{1}{1+u^2} du = \tan^{-1} u \Big|_0^{\ln 2} = \tan^{-1}(\ln 2)$

(b) $\int_0^{0.5} \frac{s}{\sqrt{1-s^4}} ds$, let $u = s^2 \Rightarrow du = 2s\,ds$

$= \int_0^{\frac{1}{4}} \frac{\frac{1}{2}}{\sqrt{1-u^2}} du = \frac{1}{2} \sin^{-1} u \Big|_0^{\frac{1}{4}} = \frac{1}{2} \sin^{-1}(\frac{1}{4})$

7. $\lim_{n\to\infty} \sum_{k=n}^{2n} \frac{n}{k^2} = \lim_{n\to\infty} \sum_{k=n}^{2n} \frac{1}{n} \cdot \frac{1}{(\frac{k}{n})^2} = \int_1^2 \frac{1}{x^2} dx = \frac{-1}{x}\Big|_1^2 = \frac{1}{2}$

8. (a) True , $\because \lim_{x\to 0} \frac{\sin x}{\cos x} = 0 \Rightarrow \sin x = o(\cos x) \Rightarrow \sin x = O(\cos x)$

(b) False , $\because \lim_{n\to\infty} \frac{\cos(2n\pi + \frac{1}{n})}{\sin(2n\pi + \frac{1}{n})} = \infty$

(c) True , $\because \lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0 \Rightarrow \ln x = o(\sqrt{x})$

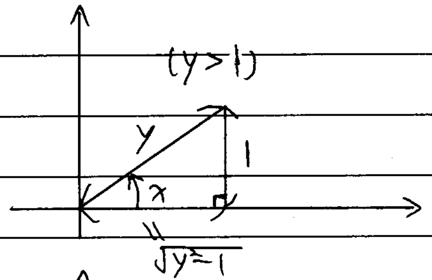
9. Show that $\frac{d\csc^{-1}y}{dy} = \frac{-1}{|y|\sqrt{y^2-1}}, \quad |y|>1$

Pf:

Let $\csc^{-1}y = x \Rightarrow \csc x = y \Rightarrow \frac{d\csc x}{dy} = 1$

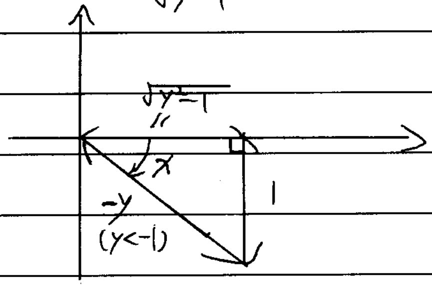
$\Rightarrow -\csc x \cdot \cot x \cdot \frac{dx}{dy} = 1$ ($\frac{dx}{dy}$ is what we want)

$\Rightarrow \frac{dx}{dy} = \frac{-1}{\csc x \cdot \cot x}$



$y>1 \Rightarrow \csc x = y$ and $\cot x = \sqrt{y^2-1}$

$y<-1 \Rightarrow \csc x = y$ and $\cot x = -\sqrt{y^2-1}$



Hence

$$\frac{d}{dy}(\csc^{-1}y) = \begin{cases} \frac{-1}{y\sqrt{y^2-1}}, & \text{if } y>1 \\ \frac{1}{y\sqrt{y^2-1}}, & \text{if } y<-1 \end{cases}$$

$$\Rightarrow \frac{d}{dy}(\csc^{-1}y) = \frac{-1}{|y|\sqrt{y^2-1}}, \quad |y|>1$$