1. $\lim_{y \to +\infty} y \sin\frac{2}{\sqrt{y}} = \lim_{y\to+\infty} \left(\frac{\sin\frac{2}{\sqrt{y}}}{\frac{2}{\sqrt{y}}} \right) \cdot 2\sqrt{y}$

$= \left(\lim_{y\to+\infty} \frac{\sin\frac{2}{\sqrt{y}}}{\frac{2}{\sqrt{y}}} \right) \left(\lim_{y\to+\infty} 2\sqrt{y} \right) = +\infty$

$\therefore$ ~~Limit~~ Limit does not exist. (6%)

2. $y=\sqrt{x+\sqrt{x}}\quad, x>0$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2}\left(x+\sqrt{x}\right)^{-\frac{1}{2}}\cdot\left(1+\frac{1}{2}x^{-\frac{1}{2}}\right)$$

3. $\tan(x+y)+\sin(x^2+y)=0$

$$\Rightarrow \left(\sec^2(x+y)\right)\left(1+\frac{dy}{dx}\right)+\left(\cos(x^2+y)\right)\left(2x+\frac{dy}{dx}\right)=0$$

$$(x,y)=(1,-1) \Rightarrow \left(\sec^2 0\right)\left(1+\frac{dy}{dx}\Big|_{(1,-1)}\right)+\left(\cos 0\right)\left(2+\frac{dy}{dx}\Big|_{(1,-1)}\right)=0$$

$$\Rightarrow \frac{dy}{dx}\Big|_{(1,-1)}=-\frac{3}{2}$$

Moreover,

$$\left(2\sec^2(x+y)\tan(x+y)\right)\left(1+\frac{dy}{dx}\right)^2+\left(\sec^2(x+y)\right)\left(\frac{d^2y}{dx^2}\right)$$
$$-\left(\sin(x^2+y)\right)\left(2x+\frac{dy}{dx}\right)^2+\left(\cos(x^2+y)\right)\left(2+\frac{d^2y}{dx^2}\right)=0$$

$$(x,y)=(1,-1) \Rightarrow (2\cdot 1\cdot 0)\left(1-\frac{3}{2}\right)^2+1\cdot\frac{d^2y}{dx^2}\Big|_{(1,-1)}-0\cdot\left(2-\frac{3}{2}\right)^2+1\cdot\left(2+\frac{d^2y}{dx^2}\Big|_{(1,-1)}\right)=0$$

$$\Rightarrow \frac{d^2y}{dx^2}\Big|_{(1,-1)}=-1$$

4. Since $\lim_{y \to L} g(y) = g(L)$

Given $\epsilon > 0$, $\exists \delta_1 > 0$ s.t. for all y, ①

$$|y - L| < \delta_1 \Rightarrow |g(y) - g(L)| < \epsilon$$

And since $\lim_{x \to c} f(x) = L$, for the $\delta_1 > 0$, $\exists \delta_2 > 0$ s.t. for all x, ②

③ $$0 < |x - c| < \delta_2 \Rightarrow |f(x) - L| < \delta_1$$

$$\Rightarrow |g(f(x)) - g(L)| < \epsilon$$ ④

Hence $\lim_{x \to c} g(f(x)) = g(L)$ (By ①~④)

5. Since $\frac{d^{10}}{dx^{10}}(x^{10}\sin x)\Big|_{x=0} = \Big[10!\,\sin x + (\quad\quad)\cdot x\Big]\Big|_{x=0} = 0$

and $\frac{d^{11}}{dx^{11}}(x^{10}\sin x)\Big|_{x=0} = \Big[11\cdot 10!\cdot\cos x + (\quad\quad)\cdot x\Big]\Big|_{x=0} = 11!$

$\Rightarrow$ The smallest $n=11$ and $\frac{d^{11}}{dx^{11}}(x^{10}\sin x)\Big|_{x=0} = 11!$

b $L(x, x_0) = f(x_0) + f'(x_0)(x - x_0)$

Let $f(x) = \sin x \Rightarrow L(x, \frac{\pi}{3}) = \sin(\frac{\pi}{3}) + \cos(\frac{\pi}{3})(x - \frac{\pi}{3}) = \frac{\sqrt{3}}{2} + \frac{1}{2}(x - \frac{\pi}{3})$

$\Rightarrow \sin(\frac{\pi}{3} - 0.01) = f(\frac{\pi}{3} - 0.01) \approx L(\frac{\pi}{3} - 0.01, \frac{\pi}{3}) = \frac{\sqrt{3} - 0.01}{2}$

Moreover, $\left| f(\frac{\pi}{3} - 0.01) - L(\frac{\pi}{3} - 0.01, \frac{\pi}{3}) \right| \leq \frac{1}{2} \max_{\xi \in [\frac{\pi}{3} - 0.01, \frac{\pi}{3}]} |f''(\xi)| \, (\frac{\pi}{3} - 0.01 - \frac{\pi}{3})^2$

$\leq \frac{1}{2} \times 10^{-4} = 5 \times 10^{-5}$

7. $f(x) = \frac{x}{\sqrt{x^2+1}} \Rightarrow f'(x) = \frac{1}{(x^2+1)^{\frac{3}{2}}}$

Given x_0, $x_{n+1} = x_n - \frac{\frac{x_n}{\sqrt{x_n^2+1}}}{\frac{1}{(x_n^2+1)^{\frac{3}{2}}}} = x_n - x_n(x_n^2+1) = -x_n^3$

Since x_k converge to x^* if $-a < x_0 < a$ and diverge otherwise

then if $x_0 = a \Rightarrow x_1 = -a$ and the iteration go back and forth

between a and $-a$

Hence $-a = -a^3 \Rightarrow a^3 - a = 0 \Rightarrow a = 0, 1, -1$

$\Rightarrow$ If $x_0 \in (-1, 1)$, x_k will converge to x^*

8. $f(x) = x^{\frac{2}{3}}(x-1)^3 \Rightarrow f'(x) = \frac{2}{3}x^{-\frac{1}{3}}(x-1)^3 + 3x^{\frac{2}{3}}(x-1)^2$

$$= \frac{1}{\sqrt[3]{x}}(x-1)^2\left(\frac{2}{3}(x-1)+3x\right)$$

$$= \frac{(11x-2)(x-1)^2}{3\sqrt[3]{x}}$$

$\Rightarrow$ critical points : $x = 0, \frac{2}{11}, 1$

$\Rightarrow \begin{cases} f' > 0 \text{ when } x > 1 \text{ or } \frac{2}{11} < x < 1 \text{ or } x < 0 \\ f' < 0 \text{ when } 0 < x < \frac{2}{11} \end{cases}$

$\Rightarrow$ f has local minimum $\left(\frac{2}{11}\right)^{\frac{2}{3}}\left(-\frac{9}{11}\right)^3$ at $x = \frac{2}{11}$

local maximum 0 at $x=0$

9.

(a) True!

Pf: Since $|f(x)| \le |x|$, $\forall x \in \mathbb{R}$

$\Rightarrow |f(0)| \le |0| = 0 \Rightarrow f(0) = 0$

then $\forall \epsilon > 0$, pick $\delta = \epsilon > 0$ s.t.

if $|x - 0| < \delta \Rightarrow |f(x) - f(0)| = |f(x)| \le |x|$

$< \delta = \epsilon$

$\Rightarrow f$ is continuous at $x = 0$

(b) True!

Pf: Since $|f(x)-(3x+2)| \le |x|^{1.5}$, $\forall x \in \mathbb{R}$

$$\Rightarrow |f(0)-2| \le |0|^{1.5}=0 \Rightarrow f(0)=2$$

$$\Rightarrow \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)-2}{x} = \lim_{x\to 0} \frac{f(x)-(3x+2)+3x}{x}$$

$$\because -|x|^{0.5} \le -\left|\frac{f(x)-(3x+2)}{x}\right| \le \frac{f(x)-(3x+2)}{x} \le \left|\frac{f(x)-(3x+2)}{x}\right| \le |x|^{0.5}$$

and $\lim_{x\to 0} |x|^{0.5} = 0$

$\therefore \lim_{x\to 0} \frac{f(x)-(3x+2)}{x} = 0$ (By Sandwich theorem)

$\therefore \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0+3=3$

$\Rightarrow$ f is differentiable at $x=0$