8.1-45

$$\int_{\frac{\pi}{2}}^{\pi}\sqrt{1+\cos 2t}\,dt = \int_{\frac{\pi}{2}}^{\pi}\sqrt{1+2\cos^2 t-1}\,dt$$

$$= \int_{\frac{\pi}{2}}^{\pi}\sqrt{2\cos^2 t}\,dt = \sqrt{2}\int_{\frac{\pi}{2}}^{\pi}|\cos t|\,dt$$

$$= \sqrt{2}\int_{\frac{\pi}{2}}^{\pi}-\cos t\,dt = -\sqrt{2}\,[\sin t]_{\frac{\pi}{2}}^{\pi}$$

$$= \sqrt{2}$$

80

$$\int ((x^2-1)(x+1))^{-\frac{2}{3}}dx \overset{\text{Let}}{=} A$$

(a)

Let $u=\frac{1}{x+1} \Rightarrow du = -\frac{1}{(x+1)^2}dx$ and $\frac{1}{1-2u}=\frac{x+1}{x-1}$

$$A=\int((x-1)(x+1)^2)^{-\frac{2}{3}}dx = \int (x-1)^{-\frac{2}{3}}(x+1)^{-\frac{4}{3}}[-(x+1)^2]du$$

$$= -\int (x-1)^{-\frac{2}{3}}(x+1)^{\frac{2}{3}}du = -\int\left(\frac{x+1}{x-1}\right)^{\frac{2}{3}}du$$

$$= -\int\left(\frac{1}{1-2u}\right)^{\frac{2}{3}}du = -\int(1-2u)^{-\frac{2}{3}}du$$

$$= \frac{3}{2}(1-2u)^{\frac{1}{3}}+C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}}+C$$

(b)

$u=\left(\frac{x-1}{x+1}\right)^k$, $k=1,\frac{1}{2},\frac{1}{3},-\frac{1}{3},-\frac{2}{3},-1$

(i) $k=1$

$u=\frac{x-1}{x+1} \Rightarrow du = \frac{2}{(x+1)^2}dx$

$$A=\int (x-1)^{-\frac{2}{3}}(x+1)^{-\frac{4}{3}}\left[\frac{1}{2}(x+1)^2\right]du$$

$$= \frac{1}{2}\int(x-1)^{-\frac{2}{3}}(x+1)^{\frac{2}{3}}du = \frac{1}{2}\int u^{-\frac{2}{3}}du = \frac{3}{2}u^{\frac{1}{3}}+C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}}+C$$

(ii) $k=\frac{1}{2}$

$$u=\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}} \Rightarrow du=\frac{1}{2}\left(\frac{x-1}{x+1}\right)^{-\frac{1}{2}}\cdot\frac{2}{(x+1)^2}dx=\left(\frac{x-1}{x+1}\right)^{-\frac{1}{2}}\frac{1}{(x+1)^2}dx$$

$$A=\int (x-1)^{-\frac{2}{3}}(x+1)^{-\frac{4}{3}}\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}}(x+1)^2\,du$$

$$=\int\left(\frac{x+1}{x-1}\right)^{\frac{2}{3}}\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}}du=\int\left(\frac{x-1}{x+1}\right)^{-\frac{1}{6}}du=\int u^{-\frac{1}{3}}du$$

$$=\frac{3}{2}u^{\frac{2}{3}}+C=\frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}}+C$$

(iii) $k=\frac{1}{3}$

$$u=\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}} \Rightarrow du=\frac{1}{3}\left(\frac{x-1}{x+1}\right)^{-\frac{2}{3}}\cdot\frac{2}{(x+1)^2}dx=\frac{2}{3}\left(\frac{x-1}{x+1}\right)^{-\frac{2}{3}}\frac{1}{(x+1)^2}dx$$

$$A=\int (x-1)^{-\frac{2}{3}}(x+1)^{-\frac{4}{3}}\cdot\frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{2}{3}}(x+1)^2\,du$$

$$=\frac{3}{2}\int\left(\frac{x+1}{x-1}\right)^{\frac{2}{3}}\left(\frac{x-1}{x+1}\right)^{\frac{2}{3}}du=\frac{3}{2}\int 1\,du=\frac{3}{2}u+C$$

$$=\frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}}+C$$

(iv) $k=-\frac{1}{3}$

$$u=\left(\frac{x-1}{x+1}\right)^{-\frac{1}{3}} \Rightarrow du=-\frac{2}{3}\left(\frac{x-1}{x+1}\right)^{-\frac{4}{3}}\frac{1}{(x+1)^2}dx$$

$$A=-\frac{3}{2}\int\left(\frac{x+1}{x-1}\right)^{\frac{2}{3}}\left(\frac{x-1}{x+1}\right)^{\frac{4}{3}}du=-\frac{3}{2}\int\left(\frac{x-1}{x+1}\right)^{\frac{2}{3}}du$$

$$=-\frac{3}{2}\int u^{-2}du=\frac{3}{2}u^{-1}+C=\frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}}+C$$

(v) $k=-\frac{2}{3}$

$$u=\left(\frac{x-1}{x+1}\right)^{-\frac{2}{3}} \Rightarrow du=-\frac{4}{3}\left(\frac{x-1}{x+1}\right)^{-\frac{5}{3}}\frac{1}{(x+1)^2}dx$$

$$A=-\frac{3}{4}\int\left(\frac{x+1}{x-1}\right)^{\frac{2}{3}}\left(\frac{x-1}{x+1}\right)^{\frac{5}{3}}du=-\frac{3}{4}\int\left(\frac{x-1}{x+1}\right)du=-\frac{3}{4}\int u^{-\frac{3}{2}}du$$

$$=-\frac{3}{4}\left(-2u^{-\frac{1}{2}}\right)+C=\frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}}+C$$

(vi) $k=-1$

$$u=\left(\frac{x-1}{x+1}\right)^{-1} \Rightarrow du=-2\left(\frac{x-1}{x+1}\right)^{-2}\frac{1}{(x+1)^2}dx$$

$$A=-\frac{1}{2}\int\left(\frac{x+1}{x-1}\right)^{\frac{2}{3}}\left(\frac{x-1}{x+1}\right)^{2}du=-\frac{1}{2}\int\left(\frac{x-1}{x+1}\right)^{\frac{4}{3}}du$$

$$=-\frac{1}{2}\int u^{-\frac{4}{3}}du=-\frac{1}{2}\left(-3u^{-\frac{1}{3}}\right)+C=\frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}}+C$$

(c)

$$u=\tan^{-1}x \Rightarrow \tan u=x \Rightarrow \sec^2 u\,du=dx$$

We will use these equalities

$$\begin{cases}\sin u+\cos u=\sin u+\sin\left(\frac{\pi}{2}-u\right)=2\sin\frac{\pi}{4}\cos\left(u-\frac{\pi}{4}\right)\\ \sin u-\cos u=\sin u-\sin\left(\frac{\pi}{2}-u\right)=2\cos\frac{\pi}{4}\sin\left(u-\frac{\pi}{4}\right)\end{cases}$$

$$A=\int(x+1)^{-2}\left(\frac{x-1}{x+1}\right)^{-\frac{2}{3}}dx=\int\frac{1}{(\tan u+1)^2}\left(\frac{\tan u-1}{\tan u+1}\right)^{-\frac{2}{3}}\frac{1}{\cos^2 u}du$$

$$=\int\frac{1}{(\sin u+\cos u)^2}\left(\frac{\sin u-\cos u}{\sin u+\cos u}\right)^{-\frac{2}{3}}du$$

$$=\int\frac{1}{2\cos^2\left(u-\frac{\pi}{4}\right)}\left[\frac{\sin\left(u-\frac{\pi}{4}\right)}{\cos\left(u-\frac{\pi}{4}\right)}\right]^{-\frac{2}{3}}du=\frac{1}{2}\int\tan^{-\frac{2}{3}}\left(u-\frac{\pi}{4}\right)\sec^2\left(u-\frac{\pi}{4}\right)du$$

$$=\frac{3}{2}\tan^{\frac{1}{3}}\left(u-\frac{\pi}{4}\right)+C=\frac{3}{2}\left[\frac{\tan u-\tan\frac{\pi}{4}}{1+\tan u\tan\frac{\pi}{4}}\right]^{\frac{1}{3}}+C=\frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}}+C$$

(d)

$$u=\tan^{-1}\sqrt{x} \Rightarrow \tan u=\sqrt{x} \Rightarrow \tan^2 u=x \Rightarrow 2\tan u\left(\sec^2 u\right)du=dx$$

$$\Rightarrow\begin{cases}x-1=\tan^2 u-1=\frac{\sin^2 u-\cos^2 u}{\cos^2 u}=\frac{1-2\cos^2 u}{\cos^2 u}\\ x+1=\tan^2 u+1=\sec^2 u=\frac{1}{\cos^2 u}\\ \cos 2u=\frac{1-\tan^2 u}{1+\tan^2 u}\end{cases}$$

$$A=\int(x-1)^{-\frac{2}{3}}(x+1)^{-\frac{4}{3}}dx=\int\frac{(1-2\cos^2 u)^{-\frac{2}{3}}}{(\cos^2 u)^{-\frac{2}{3}}}\cdot\frac{1}{(\cos^2 u)^{-\frac{4}{3}}}\cdot\frac{2\sin u}{\cos^3 u}du$$

$$= \int \frac{2\cos u}{(1-2\cos^2 u)^{\frac{2}{3}}} \sin u\, du \qquad \left(\begin{array}{l}\text{Let } w=\cos u \\ dw=-\sin u\, du\end{array}\right)$$

$$= -2\int \frac{w}{(1-2w^2)^{\frac{2}{3}}}\, dw$$

$$= -2\left(-\frac{3}{4}(1-2w^2)^{\frac{1}{3}}\right)+C = \frac{3}{2}(1-2\cos^2 u)^{\frac{1}{3}}+C$$

$$= \frac{3}{2}(-\cos 2u)^{\frac{1}{3}}+C = \frac{3}{2}\left(-\frac{1-\tan^2 u}{1+\tan^2 u}\right)^{\frac{1}{3}}+C$$

$$= \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}}+C$$

(e) $u=\tan^{-1}\left(\frac{x-1}{2}\right) \Rightarrow \tan u=\frac{x-1}{2} \Rightarrow x+1=2(\tan u+1)$

$\Rightarrow 2\sec^2 u\, du = dx$

$$A=\int (x-1)^{-\frac{2}{3}}(x+1)^{-\frac{4}{3}}dx = \int (2\tan u)^{-\frac{2}{3}}(2(\tan u+1))^{-\frac{4}{3}}\, 2\sec^2 u\, du$$

$$= \frac{1}{2}\int\left(\frac{\tan u}{1+\tan u}\right)^{-\frac{2}{3}}\frac{1}{(1+\tan u)^2}\sec^2 u\, du$$

$$= \frac{1}{2}\int\left(1-\frac{1}{1+\tan u}\right)^{-\frac{2}{3}}\frac{\sec^2 u}{(1+\tan u)^2}\, du \qquad \left(\begin{array}{l}\text{let } w=1-\frac{1}{1+\tan u} \\ dw=\frac{\sec^2 u}{(1+\tan u)^2}\, du\end{array}\right)$$

$$= \frac{1}{2}\int w^{-\frac{2}{3}}\, dw$$

$$= \frac{3}{2}w^{\frac{1}{3}}+C = \frac{3}{2}\left(1-\frac{1}{\tan u+1}\right)^{\frac{1}{3}}+C = \frac{3}{2}\left(1-\frac{2}{x+1}\right)^{\frac{1}{3}}+C$$

$$= \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}}+C$$

(f) $u=\cos^{-1}x \Rightarrow \cos u = x \Rightarrow -\sin u\, du = dx$

$$A=\int\left((\cos^2 u-1)(\cos u+1)\right)^{-\frac{2}{3}}(-\sin u)\, du$$

$$= -\int \frac{\sin u}{\sqrt[3]{(-\sin^2 u)^2(2\cos^2\frac{u}{2})^2}}\, du = -\int \frac{1}{\sqrt[3]{(\sin u)2^2\cdot\cos^4\frac{u}{2}}}\, du$$

$$= -\int \frac{1}{\sqrt[3]{2^3 \sin\frac{u}{2}\cos^5\frac{u}{2}}}\,du = -\frac{1}{2}\int \frac{1}{\left(\frac{\sin\frac{u}{2}}{\cos\frac{u}{2}}\right)^{\frac{1}{3}}\left(\cos^6\frac{u}{2}\right)^{\frac{1}{3}}}\,du$$

$$= -\frac{1}{2}\int \tan\left(\frac{u}{2}\right)^{-\frac{1}{3}} \cdot \sec^2\left(\frac{u}{2}\right) du \qquad \left(\begin{array}{l}\text{let } w = \tan\left(\frac{u}{2}\right) \\ dw = \left[\sec^2\left(\frac{u}{2}\right)\right]\frac{1}{2}\,du\end{array}\right)$$

$$= -\frac{1}{2}\int 2 w^{-\frac{1}{3}}\,dw$$

$$= -1\left(\frac{3}{2}w^{\frac{2}{3}}\right) + C = -\frac{3}{2}\left[\tan\left(\frac{u}{2}\right)\right]^{\frac{2}{3}} + C$$

$$= -\frac{3}{2}\left(\tan^2\left(\frac{u}{2}\right)\right)^{\frac{1}{3}} + C = -\frac{3}{2}\left(\frac{1-\cos u}{1+\cos u}\right)^{\frac{1}{3}} + C$$

$$= \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}} + C$$

(g) $u = \cosh^{-1} x \Rightarrow \cosh u = x \Rightarrow \sinh u\,du = dx$

$$A = \int \left((\cosh^2 u - 1)(\cosh u + 1)\right)^{-\frac{2}{3}} \sinh u\,du$$

$$= \int \left[(\sinh^2 u)^2 \left(2\cosh^2\left(\frac{u}{2}\right)\right)^2\right]^{-\frac{1}{3}} \sinh u\,du$$

$$= \int \left[(\sinh u)\left(2^2\cosh^4\left(\frac{u}{2}\right)\right)\right]^{-\frac{1}{3}} du = \int \left[\left(2\sinh\left(\frac{u}{2}\right)\cosh\left(\frac{u}{2}\right)\right)\left(2^2\cosh^4\left(\frac{u}{2}\right)\right)\right]^{-\frac{1}{3}} du$$

$$= \frac{1}{2}\int \left(\frac{\sinh\left(\frac{u}{2}\right)}{\cosh\left(\frac{u}{2}\right)} \cdot \cosh^6\left(\frac{u}{2}\right)\right)^{-\frac{1}{3}} du$$

$$= \frac{1}{2}\int \tanh^{-\frac{1}{3}}\left(\frac{u}{2}\right) \cdot \operatorname{sech}^2\left(\frac{u}{2}\right) du \qquad \left(\begin{array}{l}\text{let } w = \tanh\left(\frac{u}{2}\right) \\ dw = \left[\operatorname{sech}^2\left(\frac{u}{2}\right)\right]\frac{1}{2}\,du\end{array}\right)$$

$$= \frac{1}{2}\int 2 w^{-\frac{1}{3}}\,dw$$

$$= \frac{3}{2} w^{\frac{2}{3}} + C = \frac{3}{2}\left(\tanh^2\left(\frac{u}{2}\right)\right)^{\frac{1}{3}} + C = \frac{3}{2}\left(\frac{\cosh u - 1}{\cosh u + 1}\right)^{\frac{1}{3}} + C$$

$$= \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}} + C$$

8.2-25

$$\int e^{\sqrt{3s+9}}\,ds = \int \sqrt{3s+9}\cdot\frac{e^{\sqrt{3s+9}}}{\sqrt{3s+9}}\,ds \qquad \left(\begin{array}{l}\text{let } u=\sqrt{3s+9}\\ dv=\dfrac{e^{\sqrt{3s+9}}}{\sqrt{3s+9}}\,ds\end{array}\right)$$

$$=\frac{2}{3}\sqrt{3s+9}\,e^{\sqrt{3s+9}}-\int\frac{2}{3}e^{\sqrt{3s+9}}\cdot\frac{3}{2\sqrt{3s+9}}\,ds$$

$$=\frac{2}{3}\sqrt{3s+9}\,e^{\sqrt{3s+9}}-\frac{2}{3}e^{\sqrt{3s+9}}+C=\frac{2}{3}\left(\sqrt{3s+9}\,e^{\sqrt{3s+9}}-e^{\sqrt{3s+9}}\right)+C$$

30

$$\int z(\ln z)^2\,dz \qquad \left(\begin{array}{l}\text{let } u=(\ln z)^2\\ dv=z\,dz\end{array}\right)$$

$$=\frac{z^2}{2}(\ln z)^2-\int\frac{z^2}{2}\left(2\ln z\cdot\frac{1}{z}\right)dz=\frac{z^2}{2}(\ln z)^2-\int z\ln z\,dz$$

$$=\frac{z^2}{2}(\ln z)^2-\left(\frac{z^2}{2}\ln z-\int\frac{z^2}{2}\cdot\frac{1}{z}\,dz\right)=\frac{z^2}{2}(\ln z)^2-\frac{z^2}{2}\ln z+\frac{z^2}{4}+C$$

8.3-42

$$\int \frac{1}{\sqrt{\theta}+\sqrt[3]{\theta}}\,d\theta \qquad \left(\text{let } \theta = x^6 \Rightarrow d\theta = 6x^5 dx \right)$$

$$= \int \frac{6x^5}{x^3+x^2}\,dx$$

$$= \int \frac{6x^3}{x+1}\,dx = \int 6x^2-6x+6\,dx - 6\int \frac{1}{x+1}\,dx$$

$$= 2x^3-3x^2+6x-6\ln|x+1|+C$$

43

$$\int t\ln(t+5)\,dt \qquad \left(\begin{array}{l} u=\ln(t+5)\ ,\ dv = t\,dt \\ \Rightarrow du = \frac{1}{t+5}dt\ ,\ v=\frac{t^2}{2} \end{array} \right)$$

$$= \frac{t^2}{2}\ln(t+5) - \int \frac{t^2}{2}\frac{1}{t+5}\,dt$$

$$= \frac{t^2}{2}\ln(t+5) - \frac{1}{2}\int \frac{t^2}{t+5}\,dt = \frac{t^2}{2}\ln(t+5) - \frac{1}{2}\left(\int t-5\,dt + \int \frac{25}{t+5}\,dt\right)$$

$$= \frac{t^2}{2}\ln(t+5) - \frac{1}{2}\left(\frac{t^2}{2}-5t+25\ln|t+5|\right)+C$$

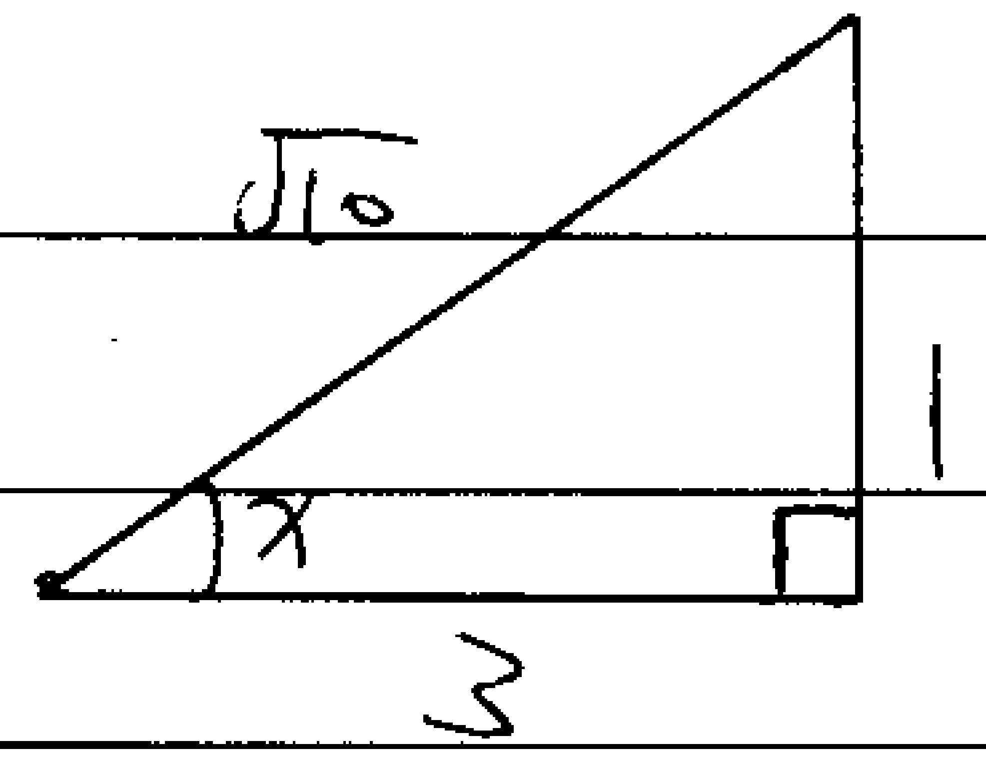
$$= \frac{t^2}{2}\ln(t+5) - \frac{t^2}{4}+\frac{5}{2}t-\frac{25}{2}\ln|t+5|+C$$

8.4 - 14

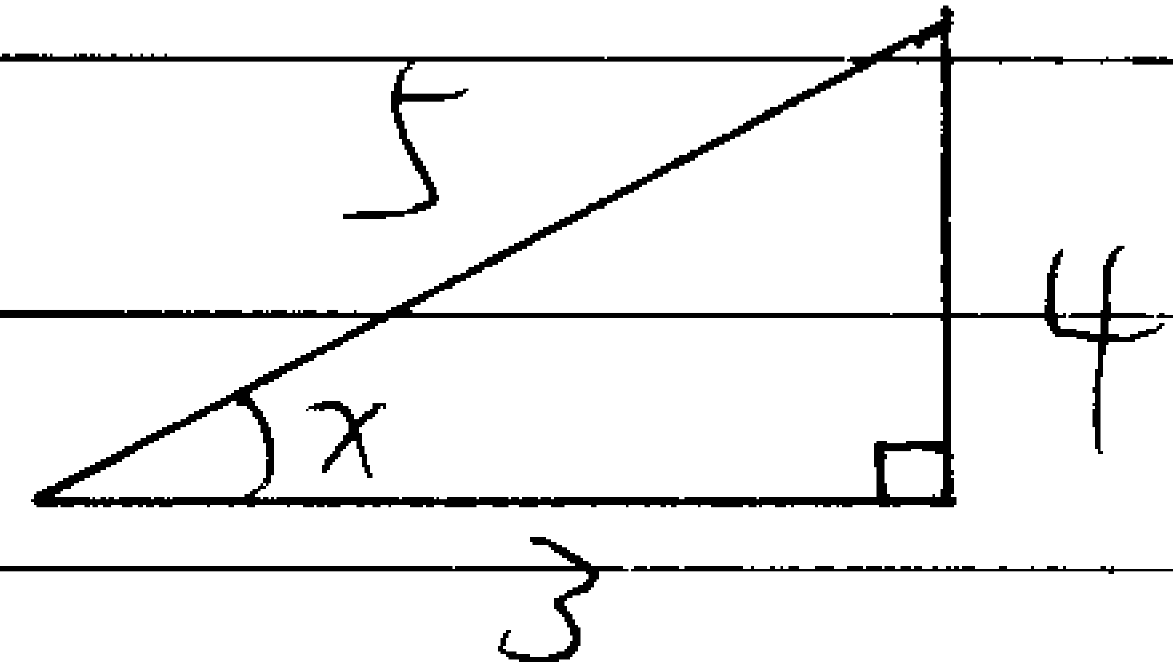
$$\int_0^{\ln 4} \frac{e^t}{\sqrt{e^{2t}+9}}\,dt \qquad \left(e^t = 3\tan x \Rightarrow e^t dt = 3\sec^2 x\,dx \right)$$

$$= \int_{t=0}^{t=\ln 4} \frac{3\sec^2 x}{\sqrt{9\tan^2 x+9}}\,dx = \int_{t=0}^{t=\ln 4} \frac{3\sec^2 x}{3\sec x}\,dx = \int_{t=0}^{t=\ln 4} \sec x\,dx$$

$$= \ln\left|\sec x + \tan x\right| \Big|_{t=0}^{t=\ln 4} \qquad \left(t=0 \Rightarrow \tan x = \frac{1}{3} \Rightarrow \sec x = \frac{\sqrt{10}}{3} \right)$$



$$= \ln 3 - \ln\left(\frac{\sqrt{10}+1}{3}\right) \qquad \left(t=\ln 4 \Rightarrow \tan x = \frac{4}{3} \Rightarrow \sec x = \frac{5}{3} \right)$$



$$= \ln\left(\frac{9}{\sqrt{10}+1}\right)$$

Ch8 - 130 $\dfrac{x^3+2}{4-x^2} = -x + \dfrac{4x+2}{4-x^2} = -x + \dfrac{\frac{5}{2}}{2-x} - \dfrac{\frac{3}{2}}{2+x}$

$$\Rightarrow \int \frac{x^3+2}{4-x^2}\,dx = -\frac{x^2}{2} - \frac{5}{2}\ln|x-2| - \frac{3}{2}\ln|x+2| + C$$

132 $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = 2\sin\sqrt{x} + C$ (Let $u=\sqrt{x}$)

134 $\displaystyle\int \frac{t-1}{\sqrt{t^2-2t}}\,dt = \sqrt{t^2-2t} + C$ (Let $u = t^2-2t$)

136 $\displaystyle\int e^t \cos e^t\,dt = \sin(e^t) + C$ (Let $u = e^t$)

138 $\displaystyle\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \tan\theta - \theta + C$ (Use $\sin^2\theta = 1-\cos^2\theta$)

140 $\displaystyle\int \frac{\cos x}{1+\sin^2 x}\,dx = \tan^{-1}(\sin x) + C$ (Let $u = \sin x$)

142 $\displaystyle\int_2^\infty \frac{1}{(x-1)^2}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{(x-1)^2}\,dx = \lim_{b\to\infty} \left[\frac{-1}{x-1}\right]_2^b = 1$

144 $\displaystyle\int \frac{d\theta}{\sqrt{1+\sqrt{\theta}}} = 4\left[\frac{(\sqrt{1+\sqrt{\theta}})^3}{3} - \sqrt{1+\sqrt{\theta}}\right] + C$ (Let $\sqrt{\theta} = \tan^2 x$)

146 $\displaystyle\int \frac{x^5}{x^4-16}\,dx = \int x\,dx + \int \frac{16x}{x^4-16}\,dx = \frac{x^2}{2} + 8\int \frac{1}{y^2-4^2}\,dy$ ($y = x^2$)

$$= \frac{x^2}{2} + \left(\int \frac{1}{y-4}\,dy - \int \frac{1}{y+4}\,dy\right) = \frac{x^2}{2} + \ln|x^2-4| - \ln|x^2+4| + C$$

148 $\displaystyle\int \frac{1}{\theta^2-2\theta+4}\,d\theta = \int \frac{1}{(\theta-1)^2+3}\,d\theta = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\theta-1}{\sqrt{3}}\right)+C$

150 $\displaystyle\int \frac{1}{(r+1)\sqrt{r^2+2r}}\,dr = \int \frac{1}{u\sqrt{u^2-1}}\,du$ (Let $u=r+1$)

$\displaystyle = \sec^{-1}|u|+C = \sec^{-1}|r+1|+C$

152 $\displaystyle\int \frac{y}{4+y^4}\,dy = \int \frac{\frac{1}{2}}{4+u^2}\,du$ (Let $u=y^2$)

$\displaystyle = \frac{1}{4}\tan^{-1}\left(\frac{u}{2}\right)+C = \frac{1}{4}\tan^{-1}\left(\frac{y^2}{2}\right)+C$

154 $\displaystyle\int \frac{1}{(x^2-1)^2}\,dx = \int \frac{1}{(x-1)^2(x+1)^2}\,dx$

$\displaystyle = \int \frac{-\frac{1}{4}}{x-1} + \frac{\frac{1}{4}}{(x-1)^2} + \frac{\frac{1}{4}}{x+1} + \frac{\frac{1}{4}}{(x+1)^2}\,dx$

$\displaystyle = \frac{1}{4}\left(-\ln|x-1| - \frac{1}{x-1} + \ln|x+1| - \frac{1}{x+1}\right)+C$

156 $\displaystyle\int (15)^{2x+1}\,dx = \frac{1}{2\ln 15}(15)^{2x+1}+C$

158 $\displaystyle\int \frac{\sqrt{1-v^2}}{v^2}\,dv = \int \frac{\cos\theta\cdot\cos\theta\,d\theta}{\sin^2\theta}$ (Let $v=\sin\theta$)

$\displaystyle = \int \frac{1-\sin^2\theta}{\sin^2\theta}\,d\theta = \int \csc^2\theta\,d\theta - \int 1\,d\theta$

$\displaystyle = -\cot\theta - \theta + C$

$\displaystyle = -\frac{\sqrt{1-v^2}}{v} - \sin^{-1}v + C$

160 $\int \ln(\sqrt{x-1})dx = \int (\ln y)\cdot 2y\,dy \quad (\text{Let } x = y^2+1)$

$= y^2\ln y - \int y\,dy \quad \left(\begin{matrix} u=\ln y, & dv = 2y\,dy \\ \Rightarrow du = \frac{1}{y}dy, & v = y^2 \end{matrix}\right)$

$= y^2\ln y - \frac{1}{2}y^2 + C$

$= (x-1)\ln(\sqrt{x-1}) - \frac{1}{2}(x-1) + C$

162 $\int \frac{x}{\sqrt{8-2x^2-x^4}}dx = \int \frac{x}{\sqrt{9-(x^2+1)^2}}dx \quad (\text{Let } y = x^2+1)$

$= \frac{1}{2}\int \frac{1}{\sqrt{9-y^2}}dy$

$= \frac{1}{2}\sin^{-1}(\frac{y}{3}) + C = \frac{1}{2}\sin^{-1}(\frac{x^2+1}{3}) + C$

164 $\int x^3 e^{x^2}dx = \frac{1}{2}\int t e^t dt \quad (\text{Let } t = x^2)$

$= \frac{1}{2}(te^t - \int e^t dt) + C \quad \left(\begin{matrix} u = t, & dv = e^t dt \\ \Rightarrow du = dt, & v = e^t \end{matrix}\right)$

$= \frac{1}{2}(x^2e^{x^2} - e^{x^2}) + C$

166 $\int_0^{\frac{\pi}{10}} \sqrt{1+\cos 5\theta}\,d\theta = \int_0^{\frac{\pi}{10}} \sqrt{2}\cos(\frac{5}{2}\theta)\,d\theta \quad (\cos 5\theta = 2\cos^2(\frac{5}{2}\theta) - 1)$

$= \frac{2\sqrt{2}}{5}\sin(\frac{5}{2}\theta)\Big|_0^{\frac{\pi}{10}}$

$= \frac{2}{5}$

168 $$\int \frac{\tan^{-1}x}{x^2}\,dx = -\frac{\tan^{-1}x}{x} + \int \frac{1}{x(1+x^2)}\,dx \qquad \left(\begin{array}{ll} u=\tan^{-1}x \ ; & dv=\frac{1}{x^2}\,dx \\ \Rightarrow du=\frac{1}{1+x^2}\,dx \ , & v=-\frac{1}{x} \end{array} \right)$$

$$= -\frac{\tan^{-1}x}{x} + \int \frac{1}{x}\,dx - \int \frac{x}{1+x^2}\,dx$$

$$= -\frac{\tan^{-1}x}{x} + \ln|x| - \frac{1}{2}\ln(1+x^2) + C$$

170 $$\int \frac{e^t}{e^{2t}+3e^t+2}\,dt = \int \frac{1}{u^2+3u+2}\,du \qquad (\text{Let } u=e^t)$$

$$= \int \frac{1}{(u+1)(u+2)}\,du = \int \frac{1}{u+1}\,du - \int \frac{1}{u+2}\,du$$

$$= \ln|u+1| - \ln|u+2| + C$$

$$= \ln\left|\frac{e^t+1}{e^t+2}\right| + C$$

172 $$\int \frac{1-\cos 2x}{1+\cos 2x}\,dx = \int \frac{2\sin^2 x}{2\cos^2 x}\,dx = \int \tan^2 x\,dx = \int \sec^2 x - 1\,dx$$

$$= \tan x - x + C$$

174 $$\int \frac{\cos x}{\sin^3 x - \sin x}\,dx = \int \frac{\cos x}{\sin x(\sin^2 x - 1)}\,dx = \int \frac{\cos x}{\sin x(-\cos^2 x)}\,dx = -\int \frac{1}{\sin x \cos x}\,dx$$

$$= -\int \frac{2}{\sin 2x}\,dx \qquad (\sin 2x = 2\sin x\cos x)$$

$$= -2\int \csc 2x\,dx = \ln|\csc 2x + \cot 2x| + C$$

176 $\int \frac{x^2-x+2}{(x^2+2)^2}dx = \int \frac{1}{x^2+2}dx - \int \frac{x}{(x^2+2)^2}dx$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{1}{2}(x^2+2)^{-1} + C$

178 $\int \tan^3 t \, dt = \int \tan t\,(\sec^2 t - 1)\,dt \qquad (\tan^2 t = \sec^2 t - 1)$

$= \int \tan t \cdot \sec^2 t \, dt - \int \tan t \, dt$

$= \frac{\tan^2 t}{2} + \ln|\cos t| + C$

180 $\int \frac{3+\sec^2 x+\sin x}{\tan x}dx = \int 3\cot x\,dx + \int \frac{\sec^2 x}{\tan x}dx + \int \cos x\,dx$

$= 3\ln|\sin x| + \ln|\tan x| + \sin x + C$

182 $\int \frac{1}{(2x-1)\sqrt{x^2-x}}dx = \int \frac{1}{(2x-1)\sqrt{(x-\frac{1}{2})^2-\frac{1}{4}}}dx = \int \frac{2}{(2x-1)\sqrt{(2x-1)^2-1}}dx \qquad (\text{Let } u = 2x-1)$

$= \int \frac{1}{u\sqrt{u^2-1}}du = \sec^{-1}|u| + C$

$= \sec^{-1}|2x-1| + C$

184 $\int e^\theta \sqrt{3+4e^\theta}\,d\theta = \frac{1}{4}\int \sqrt{u}\,du \qquad (\text{Let } u = 3+4e^\theta)$

$= \frac{1}{6}u^{\frac{3}{2}} + C$

$= \frac{1}{6}(3+4e^\theta)^{\frac{3}{2}} + C$

186 $\displaystyle\int \frac{1}{\sqrt{e^{2v}-1}}\,dv = \int \frac{\frac{1}{u}}{\sqrt{u^2-1}}\,du \quad (\text{Let } u = e^v)$

$\displaystyle = \int \frac{1}{u\sqrt{u^2-1}}\,du = \sec^{-1}|u| + C = \sec^{-1}(e^v) + C$

188 $\displaystyle\int x^5 \sin x\,dx$

differentiate ↓		integrate ↓
x^5	(+)	$\sin x$
$5x^4$	(−)	$-\cos x$
$20x^3$	(+)	$-\sin x$
$60x^2$	(−)	$\cos x$
$120x$	(+)	$\sin x$
120	(−)	$-\cos x$
0		$-\sin x$

$= -x^5\cos x + 5x^4\sin x + 20x^3\cos x$

$-60x^2\sin x - 120\cos x + 120\sin x + C$

190 $\displaystyle\int \frac{4x^3-20x}{x^4-10x^2+9}\,dx = \int \frac{1}{u}\,du \quad (\text{Let } u = x^4-10x^2+9)$

$= \ln|u| + C = \ln|x^4-10x^2+9| + C$

192 $\displaystyle\int \frac{t+1}{(t^2+2t)^{\frac{2}{3}}}\,dt = \frac{1}{2}\int \frac{1}{u^{\frac{2}{3}}}\,du \quad (\text{Let } u = t^2+2t)$

$\displaystyle = \frac{3}{2}u^{\frac{1}{3}} + C = \frac{3}{2}(t^2+2t)^{\frac{1}{3}} + C$

194 $\displaystyle\int \frac{1}{t(1+\ln t)\sqrt{(\ln t)(2+\ln t)}}\,dt = \int \frac{1}{u\sqrt{(u-1)(u+1)}}\,du \quad (\text{Let } u = 1+\ln t)$

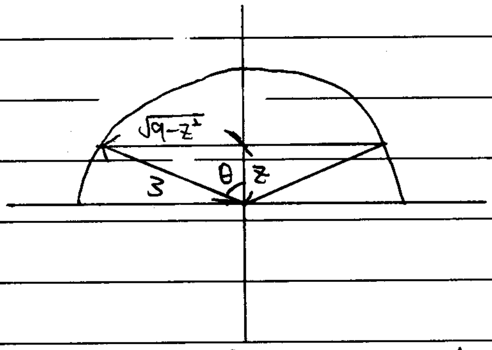
$\displaystyle = \int \frac{1}{u\sqrt{u^2-1}}\,du = \sec^{-1}|u| + C$

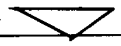
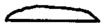
$= \sec^{-1}|1+\ln t| + C$

S8.2-extra1

Slicing in the z-direction

At height z



$A(z) =$  $-$  $=$ 

$= \pi \cdot 3^2 \cdot \frac{2\theta}{2\pi} - \frac{1}{2} \cdot z \cdot 2\sqrt{9-z^2}$

$= 9\cos^{-1}\left(\frac{z}{3}\right) - z\sqrt{9-z^2}$

$\Rightarrow \cos\theta = \frac{z}{3} \Rightarrow \theta = \cos^{-1}\left(\frac{z}{3}\right)$

$\text{Volume} = \int_0^3 9\cos^{-1}\left(\frac{z}{3}\right) - z\sqrt{9-z^2}\, dz$

(i) $\int z\sqrt{9-z^2}\, dz$, let $u = 9-z^2$, $du = -2z\, dz$

$= \int -\frac{1}{2}\sqrt{u}\, du = -\frac{1}{3}u^{\frac{3}{2}} + C = -\frac{1}{3}(9-z^2)^{\frac{3}{2}} + C$

(ii) $\int \cos^{-1}\left(\frac{z}{3}\right) dz$, let $w = \cos^{-1}\left(\frac{z}{3}\right) \Rightarrow \cos w = \frac{z}{3} \Rightarrow -\sin w\, dw = \frac{1}{3} dz$

$= \int -3w\sin w\, dw$, let $\hat{u} = w$, $dv = \sin w\, dw$

$\Rightarrow d\hat{u} = dw$, $v = -\cos w$

$= -3\left(-w\cos w + \int \cos w\, dw\right)$

$= 3w\cos w - 3\sin w + C = z\cos^{-1}\left(\frac{z}{3}\right) - \sqrt{9-z^2} + C$

$\text{Volume} = 9\left(z\cos^{-1}\left(\frac{z}{3}\right) - \sqrt{9-z^2}\right)\Big|_0^3 + \frac{1}{3}(9-z^2)^{\frac{3}{2}}\Big|_0^3$

$= 9(0-(-3)) + \left(0 - \frac{1}{3}\cdot 9^{\frac{3}{2}}\right) = 27 - 9 = 18$

$\S 8.2$-extra2 (i) From $\int \sin^{2p}(x)\cos^{2q}(x)\,dx$ to $\int \sin^{2p-2}(x)\cos^{2q}(x)\,dx$

First $\underbrace{\int \sin^{2p}(x)\cos^{2q}(x)\,dx}_{A} = \int \sin^{2p-2}(x)\cos^{2q}(x)\sin^2(x)\,dx = \int \sin^{2p-2}(x)\cos^{2q}(x)(1-\cos^2(x))\,dx$

$= \int \sin^{2p-2}(x)\cos^{2q}(x)\,dx - \underbrace{\int \sin^{2p-2}(x)\cos^{2q+2}(x)\,dx}_{B}$

By integrations by part $\begin{cases} u = \sin^{2p-1}(x) \\ dv = \cos^{2q}(x)\sin(x)\,dx \end{cases} \Rightarrow \begin{cases} du = (2p-1)\sin^{2p-2}(x)\cdot\cos(x)\,dx \\ v = \frac{-1}{2q+1}\cos^{2q+1}(x) \end{cases}$

$\Rightarrow \underbrace{\int \sin^{2p}(x)\cos^{2q}(x)\,dx}_{A} = \frac{-1}{2q+1}\sin^{2p-1}(x)\cos^{2q+1}(x) - \int \frac{-1}{2q+1}\cos^{2q+1}(x)(2p-1)\sin^{2p-2}(x)\cos(x)\,dx$

$= \frac{-1}{2q+1}\sin^{2p-1}(x)\cos^{2q+1}(x) + \frac{2p-1}{2q+1}\underbrace{\int \sin^{2p-2}(x)\cos^{2q+2}(x)\,dx}_{B}$

We have $\begin{cases} A + B = \int \sin^{2p-2}(x)\cos^{2q}(x)\,dx & \cdots ① \\ A - \frac{2p-1}{2q+1}B = \frac{-1}{2q+1}\sin^{2p-1}(x)\cos^{2q+1}(x) & \cdots ② \end{cases}$

$① \times \frac{2p-1}{2q+1} + ②$

$\Rightarrow \underbrace{\frac{2p-1}{2q+1}A + A}_{\frac{2p+2q}{2q+1}A} = \frac{2p-1}{2q+1}\int \sin^{2p-2}(x)\cos^{2q}(x)\,dx - \frac{1}{2q+1}\sin^{2p-1}(x)\cos^{2q+1}(x)$

$\Rightarrow A = -\frac{1}{2p+2q}\sin^{2p-1}(x)\cos^{2q+1}(x) + \frac{2p-1}{2p+2q}\int \sin^{2p-2}(x)\cos^{2q}(x)\,dx$

(ii) From $\int \sin^{2p}(x)\cos^{2q}(x)\,dx$ to $\int \sin^{2p}(x)\cos^{2q-2}(x)\,dx$

First $\underbrace{\int \sin^{2p}(x)\cos^{2q}(x)\,dx}_{A_1} = \int \sin^{2p}(x)\cos^{2q-2}(x)\cos^2(x)\,dx = \int \sin^{2p}(x)\cos^{2q-2}(x)(1-\sin^2(x))\,dx$

$= \int \sin^{2p}(x)\cos^{2q-2}(x)\,dx - \underbrace{\int \sin^{2p+2}(x)\cos^{2q-2}(x)\,dx}_{B_1}$

By integrations by part $\begin{cases} \tilde{u} = \cos^{2q-1}(x) \\ d\tilde{v} = \sin^{2p}(x)\cos(x)\,dx \end{cases} \Rightarrow \begin{cases} d\tilde{u} = (2q-1)\cos^{2q-2}(x)[-\sin(x)]\,dx \\ \tilde{v} = \frac{1}{2p+1}\sin^{2p+1}(x) \end{cases}$

$$\Rightarrow \underbrace{\int \sin^{2p}(x)\cos^{2q}(x)\,dx}_{A_1} = \frac{1}{2p+1}\sin^{2p+1}(x)\cos^{2q-1}(x) - \int \frac{1}{2p+1}\sin^{2p+1}(x)(2q-1)\cos^{2q-2}(x)[-\sin(x)]\,dx$$

$$= \frac{1}{2p+1}\sin^{2p+1}(x)\cos^{2q-1}(x) + \frac{2q-1}{2p+1}\underbrace{\int \sin^{2p+1}(x)\cos^{2q-2}(x)\,dx}_{B_1}$$

We have
$$\begin{cases} A_1 + B_1 = \int \sin^{2p}(x)\cos^{2q-2}(x)\,dx & \cdots ③ \\ A_1 - \frac{2q-1}{2p+1}B_1 = \frac{1}{2p+1}\sin^{2p+1}(x)\cos^{2q-1}(x) & \cdots ④ \end{cases}$$

$③ \times \frac{2q-1}{2p+1} + ④$

$$\Rightarrow \underbrace{\frac{2q-1}{2p+1}A_1 + A_1}_{=\frac{2p+2q}{2p+1}A_1} = \frac{2q-1}{2p+1}\int \sin^{2p}(x)\cos^{2q-2}(x)\,dx + \frac{1}{2p+1}\sin^{2p+1}(x)\cos^{2q-1}(x)$$

$$\Rightarrow A_1 = \frac{1}{2p+2q}\sin^{2p+1}(x)\cos^{2q-1}(x) + \frac{2q-1}{2p+2q}\int \sin^{2p}(x)\cos^{2q-2}(x)\,dx$$

5.8.2-extra3

(i) From $\int \tan^m(x)\sec^n(x)\,dx$ to $\int \tan^{m-2}(x)\sec^n(x)\,dx$

First $\underbrace{\int \tan^m(x)\sec^n(x)\,dx}_{A_2} = \int \tan^{m-2}(x)\sec^n(x)\tan^2(x)\,dx = \int \tan^{m-2}(x)\sec^n(x)\left(\sec^2(x)-1\right)dx$

$= \underbrace{\int \tan^{m-2}(x)\sec^{n+2}(x)\,dx}_{B_2} - \int \tan^{m-2}(x)\sec^n(x)\,dx$

By integration by part $\begin{cases} \hat{u} = \tan^{m-1}(x) \\ d\hat{v} = \sec^{n-1}(x)\sec(x)\tan(x)\,dx \end{cases} \Rightarrow \begin{cases} d\hat{u} = (m-1)\tan^{m-2}(x)\sec^2(x)\,dx \\ \hat{v} = \frac{1}{n}\sec^n(x) \end{cases}$

$\Rightarrow \underbrace{\int \tan^m(x)\sec^n(x)\,dx}_{A_2} = \frac{1}{n}\tan^{m-1}(x)\sec^n(x) - \int \frac{1}{n}\sec^n(x)\,(m-1)\tan^{m-2}(x)\sec^2(x)\,dx$

$= \frac{1}{n}\tan^{m-1}(x)\sec^n(x) - \frac{m-1}{n}\underbrace{\int \tan^{m-2}(x)\sec^{n+2}(x)\,dx}_{B_2}$

We have $\begin{cases} A_2 - B_2 = -\int \tan^{m-2}(x)\sec^n(x)\,dx & \cdots (5) \\ A_2 + \frac{m-1}{n}B_2 = \frac{1}{n}\tan^{m-1}(x)\sec^n(x) & \cdots (6) \end{cases}$

$(5) \times \frac{m-1}{n} + (6) \Rightarrow \underbrace{\frac{m-1}{n}A_2 + A_2}_{= \frac{m+n-1}{n}A_2} = -\frac{m-1}{n}\int \tan^{m-2}(x)\sec^n(x)\,dx + \frac{1}{n}\tan^{m-1}(x)\sec^n(x)$

$\Rightarrow A_2 = \frac{1}{m+n-1}\tan^{m-1}(x)\sec^n(x) - \frac{m-1}{m+n-1}\int \tan^{m-2}(x)\sec^n(x)\,dx$

(ii) From $\int \tan^m(x)\sec^n(x)\,dx$ to $\int \tan^m(x)\sec^{n-2}(x)\,dx$

First $\underbrace{\int \tan^m(x)\sec^n(x)\,dx}_{A_3} = \int \tan^m(x)\sec^{n-2}(x)\sec^2(x)\,dx = \int \tan^m(x)\sec^{n-2}(x)\left(1+\tan^2(x)\right)dx$

$= \int \tan^m(x)\sec^{n-2}(x)\,dx + \underbrace{\int \tan^{m+2}(x)\sec^{n-2}(x)\,dx}_{B_3}$

By integration by part $\begin{cases} \tilde{u} = \sec^{n-2}(x) \\ d\tilde{v} = \tan^m(x)\sec^2(x)\,dx \end{cases} \Rightarrow \begin{cases} d\tilde{u} = (n-2)\sec^{n-3}(x)\sec(x)\tan(x)\,dx \\ \tilde{v} = \frac{1}{m+1}\tan^{m+1}(x) \end{cases}$

$$\Rightarrow \underbrace{\int \tan^m(x)\sec^n(x)\,dx}_{A_3} = \frac{1}{m+1}\tan^{m+1}(x)\sec^{n-2}(x) - \int \frac{1}{m+1}\tan^{m+1}(x)(n-2)\sec^{n-2}(x)\tan(x)\,dx$$

$$= \frac{1}{m+1}\tan^{m+1}(x)\sec^{n-2}(x) - \frac{n-2}{m+1}\underbrace{\int \tan^{m+2}(x)\sec^{n-2}(x)\,dx}_{B_3}$$

We have $\begin{cases} A_3 - B_3 = \int \tan^m(x)\sec^{n-2}(x)\,dx & \text{---} (7) \\ A_3 + \frac{n-2}{m+1}B_3 = \frac{1}{m+1}\tan^{m+1}(x)\sec^{n-2}(x) & \text{---} (8) \end{cases}$

$$(7)\times\frac{n-2}{m+1} + (8) \Rightarrow \underset{\substack{\| \\ \frac{m+n-1}{m+1}A_3}}{\frac{n-2}{m+1}A_3 + A_3} = \frac{n-2}{m+1}\int \tan^m(x)\sec^{n-2}(x)\,dx + \frac{1}{m+1}\tan^{m+1}(x)\sec^{n-2}(x)$$

$$\Rightarrow A_3 = \frac{1}{m+n-1}\tan^{m+1}(x)\sec^{n-2}(x) + \frac{n-2}{m+n-1}\int \tan^m(x)\sec^{n-2}(x)\,dx$$