6.6

38. $\lim_{x\to 1^+} x^{\frac{1}{x-1}} = e$

40. $\lim_{x\to e^+} (\ln x)^{\frac{1}{x-e}} = e^{\frac{1}{e}}$

42. $\lim_{x\to \infty} x^{\frac{1}{\ln x}} = e$

44. $\lim_{x\to 0} (e^x + x)^{\frac{1}{x}} = e^2$

46. $\lim_{x\to 0^+} (1+\frac{1}{x})^x = 1$

54. $f(x) = \begin{cases} x+2, & x\neq 0 \\ 0, & x=0 \end{cases} \Rightarrow f'(x) = 1, \ x\neq 0$

$g(x) = \begin{cases} x+1, & x\neq 0 \\ 0, & x=0 \end{cases} \Rightarrow g'(x) = 1, \ x\neq 0$

Hence

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \frac{1}{1} = 1 \quad \text{and} \quad \lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{2}{1} = 2$$

Per-Duet

This doesn't contradict l'Hôpital's rule.

Since f and g are not continuous at $x=0$

$\Rightarrow$ f and g are not differentiable at $x=0$

$\Rightarrow$ The result doesn't contradict l'Hôpital's rule.

55.

$$f(x) = \begin{cases} \frac{9x-3\sin 3x}{5x^3}, & x \neq 0 \\ c, & x=0 \end{cases}$$

$$\lim_{x\to 0} \frac{9x-3\sin 3x}{5x^3} = \lim_{x\to 0} \frac{9-9\cos 3x}{15x^2} = \lim_{x\to 0} \frac{27\sin 3x}{30x} = \lim_{x\to 0} \frac{27}{10}\frac{\sin 3x}{3x} = \frac{27}{10}$$

$\Rightarrow c = \frac{27}{10}$

Since $\lim_{x\to 0} f(x) = \frac{27}{10} = f(0)$ (if we let $c=\frac{27}{10}$)

$\Rightarrow$ f is continuous at $x=0$

6.7-8

Since $e > 2$

$$\Rightarrow \lim_{x\to\infty} \frac{e^x}{2^x} = \lim_{x\to\infty} \left(\frac{e}{2}\right)^x = \infty$$

and $\lim_{x\to\infty} \frac{x^2}{(\ln 2)^x} = \infty$, $(\ln 2 < 1)$

$$\lim_{x\to\infty} \frac{x^2}{2^x} = \lim_{x\to\infty} \frac{2x}{2^x \ln 2} = \lim_{x\to\infty} \frac{2}{2^x (\ln 2)^2} = 0$$

Hence from slowest growing to fastest as $x \to \infty$:

$$\Rightarrow (\ln 2)^x,\ x^2,\ 2^x,\ e^x$$

12.

(a) T (b) T (c) F (d) T (e) T (f) T (g) T (h) F

18.

Since $\lim_{n\to\infty} \frac{n}{\sqrt{n}\log_2 n} = \lim_{n\to\infty} \ln 2 \frac{\sqrt{n}}{\ln n} = \lim_{n\to\infty} \ln 2 \frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}}$

$$= \lim_{n\to\infty} [\ln 2] \frac{\sqrt{n}}{2} = \infty$$

and $\lim_{n\to\infty} \frac{\sqrt{n}\log_2 n}{(\log_2 n)^2} = \lim_{n\to\infty} \frac{\sqrt{n}}{\log_2 n} = \lim_{n\to\infty} \ln 2 \frac{\sqrt{n}}{\ln n} = \lim_{n\to\infty} \ln 2 \frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}}$

$$= \lim_{n\to\infty} [\ln 2] \frac{\sqrt{n}}{2} = \infty$$

Hence

n faster than $\sqrt{n}\log_2 n$ faster than $(\log_2 n)^2$.

$\Rightarrow$ The algorithm with steps order of $(\log_2 n)^2$ is the most efficient in the long run.

Per-Duet

56.6-extra 1

Let $F(x) = \begin{vmatrix} f(b)-f(a) & f(x)-f(a) \\ g(b)-g(a) & g(x)-g(a) \end{vmatrix} = [f(b)-f(a)][g(x)-g(a)] - [g(b)-g(a)][f(x)-f(a)]$

$\because$ f and g are continuous on $[a,b]$ and differentiable on (a,b)

$\therefore$ F is continuous on $[a,b]$ and differentiable on (a,b)

$\because F(a)=F(b)=0$, by Rolle's theorem, $\exists\, c \in (a,b)$ s.t. $F'(c)=0$

where $F'(x) = \begin{vmatrix} [f(b)-f(a)]' & f(x)-f(a) \\ [g(b)-g(a)]' & g(x)-g(a) \end{vmatrix} + \begin{vmatrix} f(b)-f(a) & [f(x)-f(a)]' \\ g(b)-g(a) & [g(x)-g(a)]' \end{vmatrix}$

$= \begin{vmatrix} f(b)-f(a) & f'(x) \\ g(b)-g(a) & g'(x) \end{vmatrix}$

Hence $\begin{vmatrix} f(b)-f(a) & f'(c) \\ g(b)-g(a) & g'(c) \end{vmatrix} = 0$

sb.6-extra 2

Since f and g are differentiable on an open interval containing a.

$\Rightarrow \exists\, \delta > 0$ s.t. f and g are differentiable on $[a-\delta, a+\delta]$

By Cauchy's Mean Value Theorem, $\forall\, x \in (a-\delta, a+\delta)$, $\exists\, c$ between a & x

such that $\begin{vmatrix} f(x)-f(a) & f'(c) \\ g(x)-g(a) & g'(c) \end{vmatrix} = 0 \Rightarrow [f(x)-f(a)]g'(c) = [g(x)-g(a)]f'(c)$

Since $g'(x) \neq 0$ if $x \neq a \Rightarrow g(x) \neq g(a)$ if $x \neq a$ (or by M.V.T., we would get a contradiction)

$$\Rightarrow \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c)}{g'(c)}$$, $\forall\, x \in (a-\delta, a+\delta)$ and c is between a & x

Hence $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim\limits_{x\to a} \frac{f'(c)}{g'(c)} = \lim\limits_{c\to a} \frac{f'(c)}{g'(c)}$

$\because c$ is between a & x

56.9-extra1 Show that $\frac{d\csc^{-1}y}{dy} = \frac{-1}{|y|\sqrt{y^2-1}}$, $|y|>1$

Pf:

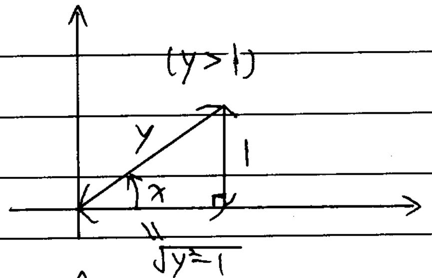
Let $\csc^{-1}y = x \Rightarrow \csc x = y \Rightarrow \frac{d\csc x}{dy} = 1$

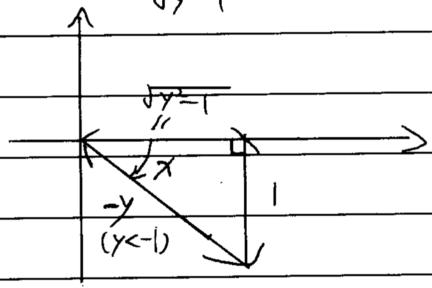
$\Rightarrow -\csc x \cdot \cot x \cdot \frac{dx}{dy} = 1$ ($\frac{dx}{dy}$ is what we want)

$\Rightarrow \frac{dx}{dy} = \frac{-1}{\csc x \cdot \cot x}$

$y>1 \Rightarrow \csc x = y$ and $\cot x = \sqrt{y^2-1}$

$y<-1 \Rightarrow \csc x = y$ and $\cot x = -\sqrt{y^2-1}$





∴ Hence

$$\frac{d}{dy}(\csc^{-1}y) = \begin{cases} \frac{-1}{y\sqrt{y^2-1}}, & \text{if } y>1 \\ \frac{1}{y\sqrt{y^2-1}}, & \text{if } y<-1 \end{cases}$$

$$\Rightarrow \frac{d}{dy}(\csc^{-1}y) = \frac{-1}{|y|\sqrt{y^2-1}}, \quad |y|>1$$