4.6-23(b)

Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

$\Rightarrow f'(x) = nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$

If $\exists$ a, b s.t. $f(a) = 0 = f(b)$, then by Rolle's Theorem,

there exists $c \in (a,b)$ s.t. $f'(c) = 0$ #

25. If $f(0)=f(1)$

then by Mean Value Theorem, there exist $c\in(0,1)$ such that

$f'(c)=\frac{f(1)-f(0)}{1-0}=\frac{0}{1}=0$ $(\rightarrow\leftarrow)$

Hence $f(0)\neq f(1)$

26.

W.L.O.G. may assume $a<b$, since $\sin x$ is differentiable on $\mathbb{R}$,

by Mean Value Theorem, $\exists\, c\in(a,b)$: such that

$\cos c=\frac{\sin b-\sin a}{b-a}\Rightarrow|\sin b-\sin a|=|b-a||\cos c|$

(Here, $|\cos c|\leq 1$) $\leq|b-a|$

28.

Let $h(x)=f(x)-g(x)$ is differentiable on $[a,b]$

$\Rightarrow h(a)=h(b)=0$

By Rolle's Theorem, $\exists\, c\in(a,b)$ s.t. $h'(c)=0$

$\Rightarrow f'(c)-g'(c)=0\Rightarrow f'(c)=g'(c)$ for some $c\in(a,b)$

33.

$f'(x)=2$ for all $x\Rightarrow\forall\, a,b\in\mathbb{R}$, $\frac{f(b)-f(a)}{b-a}=f'(c)=2$, for some $c\in(a,b)$

slope

$\Rightarrow f$ is a line function and $f(0)=5$

$\Rightarrow f(x)=5+2x$

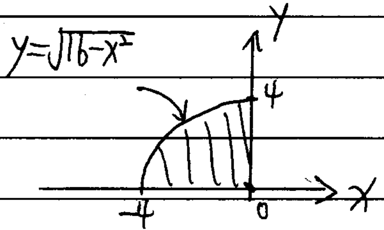
5.2-34 $\|P\| = 1.1$

36 $\lim_{\|P\| \to 0} \sum_{k=1}^{n} 2c_k^3 \Delta x_k = \int_{-1}^{0} 2x^3 \, dx$

38 $\lim_{\|P\| \to 0} \sum_{k=1}^{n} \left(\frac{1}{c_k}\right) \Delta x_k = \int_{1}^{4} \frac{1}{x} \, dx$

46 $\int_{-4}^{0} \sqrt{16 - x^2} \, dx$

$= \pi \cdot 4^2 \cdot \frac{1}{4} = 4\pi$



S5.2-extra1 Evaluate $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{1}{n}\sqrt{1-\left(\frac{k}{n}\right)^2}$

Let $f(x)=\sqrt{1-x^2}$ on $[-1,1]$

$$\begin{cases} c_k = \left(\frac{1-0}{n}\right)k = \frac{k}{n} \\ \Delta x_k = \frac{1-0}{n} = \frac{1}{n} \end{cases}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\sqrt{1-\left(\frac{k}{n}\right)^2} = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\cdot\frac{1}{n}$$

$$\left(\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \int_a^b f(x)\,dx\right)$$

$$= \int_0^1 f(x)\,dx = \int_0^1 \sqrt{1-x^2}\,dx$$

(Similar to 5.2-45) $= \pi\cdot 1^2\cdot\frac{1}{4} = \frac{\pi}{4}$

5.5.2-extra2 (i) $\ell=0$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\left(\frac{k}{n}\right)^{0}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}=\lim_{n\to\infty}1=1=\frac{1}{0+1}\text{ , hold.}$$

(ii) Assume $0\le \ell\le m-1$ holds , $m\in\mathbb{N}$

i.e. $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\left(\frac{k}{n}\right)^{\ell}=\frac{1}{\ell+1}$, $0\le \ell\le m-1$

then when $\ell=m$, want to show that $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\left(\frac{k}{n}\right)^{m}=\frac{1}{m+1}$

Pf:

$$\because\ (k+1)^{m+1}-k^{m+1}=(m+1)k^{m}+\binom{m+1}{2}k^{m-1}+\cdots+(m+1)k+1$$

$$\therefore\ \underbrace{\sum_{k=1}^{n}\left[(k+1)^{m+1}-k^{m+1}\right]}_{=(n+1)^{m+1}-1}=(m+1)\sum_{k=1}^{n}k^{m}+\binom{m+1}{2}\sum_{k=1}^{n}k^{m-1}+\cdots+(m+1)\sum_{k=1}^{n}k+\sum_{k=1}^{n}1$$

$$\Rightarrow (m+1)\sum_{k=1}^{n}k^{m}=(n+1)^{m+1}-1-\binom{m+1}{2}\sum_{k=1}^{n}k^{m-1}-\cdots-(m+1)\sum_{k=1}^{n}k-n$$

$$\Rightarrow \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\left(\frac{k}{n}\right)^{m}=\frac{1}{m+1}\lim_{n\to\infty}\frac{1}{n^{m+1}}(m+1)\sum_{k=1}^{n}k^{m}$$

$$=\frac{1}{m+1}\lim_{n\to\infty}\frac{1}{n^{m+1}}\left[(n+1)^{m+1}-1-\binom{m+1}{2}\sum_{k=1}^{n}k^{m-1}-\cdots-(m+1)\sum_{k=1}^{n}k-n\right]$$

$$=\frac{1}{m+1}\left[\lim_{n\to\infty}\frac{(n+1)^{m+1}-1}{n^{m+1}}-\binom{m+1}{2}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{n}\left(\frac{k}{n}\right)^{m-1}-\cdots\right.$$

$$\left.\cdots-(m+1)\lim_{n\to\infty}\frac{1}{n^{m-1}}\sum_{k=1}^{n}\frac{1}{n}\left(\frac{k}{n}\right)-\lim_{n\to\infty}\frac{n}{n^{m+1}}\right]$$

$$=\frac{1}{m+1}\left[1-0-\cdots-0-0\right]=\frac{1}{m+1}$$

By mathematical induction , $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\left(\frac{k}{n}\right)^{\ell}=\frac{1}{\ell+1}$, for any integer $\ell\ge 0$