HW 02

25–30 $f(x)=\sqrt{1-5x}$, $x_0=-3$, $\epsilon=0.5$

$\lim_{x\to-3}\sqrt{1-5x} = 4$

$|\sqrt{1-5x}-4|<\epsilon=0.5 \iff -\frac{1}{2}<\sqrt{1-5x}-4<\frac{1}{2}$

$\iff \frac{7}{2}<\sqrt{1-5x}<\frac{9}{2}$

$\iff \frac{49}{4}<1-5x<\frac{81}{4}$

$\iff \frac{45}{4}<-5x<\frac{77}{4} \iff -\frac{77}{20}<x<-\frac{9}{4}$

$\iff -\frac{17}{20}<x-(-3)<\frac{3}{4}=\frac{15}{20}$

Take $\delta=\min\left\{\left|\frac{-17}{20}\right|,\left|\frac{15}{20}\right|\right\}=\frac{15}{20}=\frac{3}{4}$

32

$f(x)=\frac{4}{x}$, $x_0=\frac{1}{2}$, $\epsilon=0.04$

$\lim_{x\to\frac{1}{2}}\frac{4}{x}=8$

$\left|\frac{4}{x}-8\right|<\epsilon=0.04 \iff -0.04<\frac{4}{x}-8<0.04 \iff 7.96<\frac{4}{x}<8.04$

$\iff \frac{1}{2.01}<x<\frac{1}{1.99} \iff \frac{1}{2.01}-\frac{1}{2}<x-\frac{1}{2}<\frac{1}{1.99}-\frac{1}{2}$

Take $\delta=\min\left\{\left|\frac{1}{2.01}-\frac{1}{2}\right|,\left|\frac{1}{1.99}-\frac{1}{2}\right|\right\}$

38.

$$f(x) = \frac{|x-5|}{x-5} = \begin{cases} 1 & , x>5 \\ -1 & , x<5 \end{cases}$$

(a) $\epsilon = 4$, $-3 < f(x) < 5 \Rightarrow x \in \mathbb{R} \setminus \{5\}$

(b) $\epsilon = 2$, $-1 < f(x) < 3 \Rightarrow x > 5$

(c) $\epsilon = 1$, $0 < f(x) < 2 \Rightarrow x > 5$

(d) $\epsilon = \frac{1}{2}$, $\frac{1}{2} < f(x) < \frac{3}{2} \Rightarrow x > 5$

41.

Let $f(x) = \begin{cases} \sin x & , x \neq \frac{\pi}{2} \\ 2 & , x = \frac{\pi}{2} \end{cases}$

Consider the point $x = \frac{\pi}{2}$, $f(x)$ increases to 1 as $x \to \frac{\pi}{2}^+$
or $x \to \frac{\pi}{2}^-$

So $f(x)$ gets closer to 2, but $\lim_{x \to \frac{\pi}{2}} f(x) = 1$

Since given $\epsilon > 0$, $\forall \delta > 0$, $\exists x \in (\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta)$

such that $|f(x) - 2| > 1$

(No matter how closer we take on the domain, there's always

a distance between $\lim_{x \to \frac{\pi}{2}} f(x)$ & $f(\frac{\pi}{2})$)

Chap 2-74 $\lim_{x\to 0} |x| = 0$

$\big| |x| - 0 \big| < \epsilon \Rightarrow |x| < \epsilon \Rightarrow |x - 0| < \epsilon$

So, given $\epsilon > 0$, we take $\delta = \epsilon$

then for all x, if $|x - 0| < \delta \Rightarrow |f(x) - 0| = \big||x| - 0\big| = |x| < \delta = \epsilon$

.75 $f(x) = 5x - 10$, $x_0 = 3$, $\epsilon = 0.05$

$\lim_{x\to 3} 5x - 10 = 5$

$|5x - 10 - 5| < 0.05 \Leftrightarrow -0.05 < 5x - 15 < 0.05$

$\Leftrightarrow 14.95 < 5x < 15.05$

$\Leftrightarrow 2.99 < x < 3.01 \Leftrightarrow -0.01 < x - 3 < 0.01$

We take $\delta = 0.01$

52.5-extra 1

(a) $\lim_{x\to c^+} f(x)=L$: Given $\epsilon>0$, $\exists\, \delta>0$, s.t. for all x, $c<x<c+\delta$
$\Rightarrow |f(x)-L|<\epsilon$

(b) $\lim_{x\to c} f(x)=\infty$: Given $M\in\mathbb{R}$, $\exists\, \delta>0$, s.t. for all x, $0<|x-c|<\delta$
$\Rightarrow f(x)>M$

(c) $\lim_{x\to -\infty} f(x)=L$: Given $\epsilon>0$, $\exists\, M\in\mathbb{R}$ s.t. for all x, $x<M$
$\Rightarrow |f(x)-L|<\epsilon$

S2.5-extra2 If $f(x)$ and $g(x)$ are both continuous at $x=c$

$\Rightarrow$ Given $\epsilon>0$, $\exists\, \delta_1>0$ s.t. for all x, $0<|x-c|<\delta_1 \Rightarrow |f(x)-f(c)|<\frac{\epsilon}{2}$

$\exists\, \delta_2>0$ s.t. for all x, $0<|x-c|<\delta_2 \Rightarrow |g(x)-g(c)|<\frac{\epsilon}{2}$

take $\delta=\min\{\delta_1,\delta_2\}$, then for all x, $0<|x-c|<\delta$

$$\Rightarrow |f(x)+g(x)-(f(c)+g(c))| = |(f(x)-f(c))+(g(x)-g(c))|$$

$$\leq |f(x)-f(c)|+|g(x)-g(c)| < \epsilon$$

$$\& \; |f(x)-g(x)-(f(c)-g(c))| = |(f(x)-f(c))-(g(x)-g(c))|$$

$$\leq |f(x)-f(c)|+|g(x)-g(c)| < \epsilon$$

Hence both $f(x)+g(x)$ & $f(x)-g(x)$ are continuous at $x=c$

2.5-extra3

Since $\lim_{y \to L} g(y) = g(L)$

Given $\epsilon > 0$, $\exists \delta_1 > 0$ s.t. for all y,

$|y - L| < \delta_1 \Rightarrow |g(y) - g(L)| < \epsilon$

And since $\lim_{x \to c} f(x) = L$, for the $\delta_1 > 0$, $\exists \delta_2 > 0$ s.t. for all x,

$0 < |x - c| < \delta_2 \Rightarrow |f(x) - L| < \delta_1$

$\Rightarrow |g(f(x)) - g(L)| < \epsilon$

Hence $\lim_{x \to c} g(f(x)) = g(L)$ (By ①~④)

5.2.5-extra4

Let $f:(0,1)\to\mathbb{R}$ be defined as

$$f(x)=\begin{cases}\frac{1}{p}, & \text{if } x=\frac{q}{p}, \quad p,q\in\mathbb{N}, (p,q)=1\\ 0, & \text{otherwise}\end{cases}$$

(i) f is discontinuous at $x\in(0,1)\cap\mathbb{Q}$

Pf:

If $x\in(0,1)\cap\mathbb{Q} \Rightarrow x=\frac{q}{p}$, for some $p,q\in\mathbb{N}$, $(p,q)=1$

and hence $f(x)=\frac{1}{p}$

For ϵ, $0<\epsilon<\frac{1}{p}$, $\forall \delta>0$, $\exists y\in(0,1)\setminus\mathbb{Q}$ and $y\in B(x,\delta)$

such that $|f(x)-f(y)|=\frac{1}{p}>\epsilon$

$\Rightarrow f$ is discontinuous at x

(ii) f is continuous at $x\in(0,1)\setminus\mathbb{Q}$

Pf:

If $x\in(0,1)\setminus\mathbb{Q} \Rightarrow f(x)=0$

Given $\epsilon>0$, $\exists p\in\mathbb{N}$ s.t. $\frac{1}{p}<\epsilon$, pick $\delta=\min\{|x-\frac{1}{2}|, |x-\frac{1}{3}|, |x-\frac{2}{3}|, \cdots\cdots,$
$|x-\frac{1}{p}|, |x-\frac{2}{p}|, \cdots, |x-\frac{p-1}{p}|\}>0$

then if $|y-x|<\delta \Rightarrow f(y)<\frac{1}{p} \Rightarrow |f(x)-f(y)|<\epsilon$

Hence f is continuous at x