1. (i) $\frac{dy}{dx} = e^{x-y} \Rightarrow \int e^{y}\, dy = \int e^{x} dx$

$\Rightarrow e^{y} = e^{x} + c$, for some constant c

$\Rightarrow y = \ln(e^{x}+c)$

(ii) $x\frac{dy}{dx} + y = \sin x \Rightarrow \frac{d}{dx}(xy) = \sin x$

$\Rightarrow xy = \int \sin x\, dx = -\cos x + c$
for some constant c

$\Rightarrow y = \frac{1}{x}(-\cos x + c)$

2. (i) Volume $= \int_{-1}^{1} \pi [(2+\sqrt{1-y^2})^2 - 2^2]\, dy$

$= \pi\int_{-1}^{1} 4\sqrt{1-y^2} + 1 - y^2\, dy = 4\pi \cdot \frac{\pi}{2} + \pi [y - \frac{y^3}{3}]\Big|_{-1}^{1}$

$= 2\pi^2 + \frac{4}{3}\pi$

or

Volume $= \int_{2}^{3} 2\pi x \left(\sqrt{1-(x-2)^2} - (-\sqrt{1-(x-2)^2})\right) dx$

$= 4\pi \int_{2}^{3} x\sqrt{1-(x-2)^2}\, dx$, let $x-2 = \sin\theta$
$\Rightarrow dx = \cos\theta\, d\theta$

$= 4\pi \int_{0}^{\frac{\pi}{2}} (\sin\theta + 2)\cos^2\theta\, d\theta$

$= 4\pi \left[\int_{0}^{\frac{\pi}{2}} \sin\theta \cos^2\theta\, d\theta + \int_{0}^{\frac{\pi}{2}} 2\cos^2\theta\, d\theta\right]$

$= 4\pi \left[-\frac{1}{3}\cos^3\theta\Big|_{0}^{\frac{\pi}{2}} + \int_{0}^{\frac{\pi}{2}} \cos 2\theta + 1\, d\theta\right]$

$= \frac{4}{3}\pi + 4\pi[\frac{1}{2}\sin 2\theta + \theta]_{0}^{\frac{\pi}{2}} = 2\pi^2 + \frac{4}{3}\pi$

2. (ii) Surface area generated by $x=2+\sqrt{1-y^2}$, $-1\le y\le 1$

$$\Rightarrow \frac{dx}{dy}=\frac{-y}{\sqrt{1-y^2}}$$

$$\int_{-1}^{1} 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=\int_{-1}^{1}2\pi\left[2+\sqrt{1-y^2}\right]\sqrt{1+\frac{y^2}{1-y^2}}\,dy$$

$$=2\pi\int_{-1}^{1}\frac{2}{\sqrt{1-y^2}}+1\,dy=2\pi\left[2\sin^{-1}y+y\right]\Big|_{-1}^{1}=2\pi\left[2\pi+2\right]=4\pi^2+4\pi$$

Surface area generated by $x=2$, $-1\le y\le 1$

$$\Rightarrow \frac{dx}{dy}=0$$

$$\int_{-1}^{1}2\pi\cdot 2\cdot\sqrt{1+0^2}\,dy=4\pi\int_{-1}^{1}1\,dy=8\pi$$

Surface area $=(4\pi^2+4\pi)+8\pi=4\pi^2+12\pi$

3. $$\frac{x^7}{(1-x^4)^2}=\frac{x^7}{[(1-x)(1+x)(1+x^2)]^2}=\frac{x^7}{(1-x)^2(1+x)^2(1+x^2)^2}$$

$$=\frac{A}{1-x}+\frac{B}{(1-x)^2}+\frac{C}{1+x}+\frac{D}{(1+x)^2}+\frac{Ex+F}{1+x^2}+\frac{Gx+H}{(1+x^2)^2}$$

for some constant A,B,C,D,E,F,G,H

4 (1) $\int\frac{1}{2+\sin x}dx$, let $z=\tan\frac{x}{2}\Rightarrow dx=\frac{2}{1+z^2}dz$

$\sin x=\frac{2z}{1+z^2}$

$$=\int\frac{1}{2+\frac{2z}{1+z^2}}\cdot\frac{2}{1+z^2}dz$$

$$=\int\frac{1}{z^2+z+1}dz=\int\frac{1}{(z+\frac{1}{2})^2+\frac{3}{4}}dz$$, let $z+\frac{1}{2}=\frac{\sqrt{3}}{2}\tan\theta$

$dz=\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$

$$=\int\frac{\frac{\sqrt{3}}{2}\sec^2\theta}{\frac{3}{4}\sec^2\theta}d\theta=\frac{2}{\sqrt{3}}\int 1\,d\theta=\frac{2}{\sqrt{3}}\theta+C$$

$$=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}(z+\tfrac{1}{2})\right)+C=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}(\tan\tfrac{x}{2}+\tfrac{1}{2})\right)+C$$

(2) $\int e^x \sin x \, dx \qquad u = e^x,\ dv = \sin x\, dx \Rightarrow du = e^x dx,\ V = -\cos x$

$= -e^x \cos x - \int -\cos x \, e^x \, dx$

$= -e^x \cos x + \int e^x \cos x \, dx \qquad \tilde{u} = e^x,\ d\tilde{v} = \cos x\, dx \Rightarrow d\tilde{u} = e^x dx$

$\tilde{v} = \sin x$

$= -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$

$\Rightarrow \int e^x \sin x \, dx = \frac{1}{2}(e^x \sin x - e^x \cos x) + C$

(3) $\int \frac{1}{\sqrt{4x - x^2}} dx = \int \frac{1}{\sqrt{4-(x-2)^2}} dx \qquad$ let $x-2 = 2\sin\theta$

$dx = 2\cos\theta \, d\theta$

$= \int \frac{2\cos\theta}{2\cos\theta} d\theta$

$= \theta + C = \sin^{-1}\left(\frac{x-2}{2}\right) + C$

(14) $\int_1^2 \frac{1}{e^x - e^{-x}} dx \qquad$ let $u = e^x$, $du = e^x dx \Rightarrow dx = \frac{1}{u} du$

$= \int_e^{e^2} \frac{1}{u - \frac{1}{u}} \cdot \frac{1}{u} du = \int_e^{e^2} \frac{1}{u^2 - 1} du = \int_e^{e^2} \frac{\frac{1}{2}}{u-1} - \frac{\frac{1}{2}}{u+1} du$

$= \frac{1}{2}\left[\ln|u-1| - \ln|u+1| \right]_e^{e^2} = \frac{1}{2}\left[\ln \frac{e^2-1}{e^2+1} - \ln \frac{e-1}{e+1} \right]$

(15) $\int_0^{\frac{\pi}{4}} \tan^3 x \sec^3 x \, dx \qquad$ let $u = \sec x$, $du = \sec x \tan x \, dx$

$= \int_1^{\sqrt{2}} (u^2 - 1) u^2 \, du = \frac{u^5}{5} - \frac{u^3}{3} \Big|_1^{\sqrt{2}} = \frac{2}{15}\sqrt{2} + \frac{2}{15}$

(16) $\int x^2 e^{-x}\,dx$ $\quad u=x^2,\ dv=e^{-x}dx \Rightarrow du=2x\,dx,\ v=-e^{-x}$

$= -x^2e^{-x}-\int -2xe^{-x}dx$

$= -x^2e^{-x}+\int 2xe^{-x}dx$ $\quad \tilde{u}=2x,\ d\tilde{v}=e^{-x}dx \Rightarrow d\tilde{u}=2dx,\ \tilde{v}=-e^{-x}$

$= -x^2e^{-x}+\left[-2xe^{-x}-\int -2e^{-x}dx\right]$

$= -x^2e^{-x}-2xe^{-x}+2\int e^{-x}dx = -x^2e^{-x}-2xe^{-x}-2e^{-x}+C$

$\lim_{M\to\infty}\int_0^M x^2e^{-x}dx = \lim_{M\to\infty}\left[-M^2e^{-M}-2Me^{-M}-2e^{-M}+2\right]=2$

Since $\lim_{M\to\infty} M^2e^{-M} = \lim_{M\to\infty}\frac{M^2}{e^M} = \lim_{M\to\infty}\frac{2M}{e^M} = \lim_{M\to\infty}\frac{2}{e^M}=0$

similarly, $\lim_{M\to\infty} 2Me^{-M}=0$

(17) $\int_0^1 \frac{1}{\sqrt{1+e^x}}dx$ $\quad$ let $u^2=1+e^x$, $2u\,du=e^x dx \Rightarrow \frac{2u}{u^2-1}du=dx$

$= \int_{\sqrt{2}}^{\sqrt{1+e}} \frac{1}{u}\frac{2u}{u^2-1}du = \int_{\sqrt{2}}^{\sqrt{1+e}} \frac{1}{u-1}-\frac{1}{u+1}du = \ln|u-1|-\ln|u+1|\Big|_{\sqrt{2}}^{\sqrt{e+1}}$

$= 1-2\ln(\sqrt{e+1}+1)+2\ln(\sqrt{2}+1)$