Midterm II

1. $y=\frac{x^3+2x-2}{x+1}=x^2-x+3-\frac{5}{x+1} \Rightarrow \frac{dy}{dx}=2x-1+\frac{5}{(x+1)^2}$

$$\Rightarrow \frac{d^2y}{dx^2}=2-\frac{10}{(x+1)^3}$$

$\lim_{x\to\infty}\left(x^2-x+3-\frac{5}{x+1}\right)=\infty$ and $\lim_{x\to-\infty}\left(x^2-x+3-\frac{5}{x+1}\right)=\infty$

$\Rightarrow$ when $x\to\infty$, x^2-x+3 dominates

when $x\to-\infty$, x^2-x+3 dominates

$\lim_{x\to-1^+}\left(x^2-x+3-\frac{5}{x+1}\right)=-\infty$, $\lim_{x\to-1^-}\left(x^2-x+3-\frac{5}{x+1}\right)=\infty$

$\Rightarrow$ When x close to -1, $-\frac{5}{x+1}$ dominates

and $x=-1$ is a vertical asymptote

$\frac{dy}{dx}=0 \Rightarrow 2x-1+\frac{5}{(x+1)^2}=0$ and $x\neq-1$

$\Rightarrow 2x^3+3x^2+4=0 \Rightarrow (x+2)(2x^2-x+2)=0$

$\because$ $2x^2-x+2=2\left((x-\frac{1}{4})^2+\frac{15}{16}\right)>0 \Rightarrow \frac{dy}{dx}=0$ implies $x=-2$

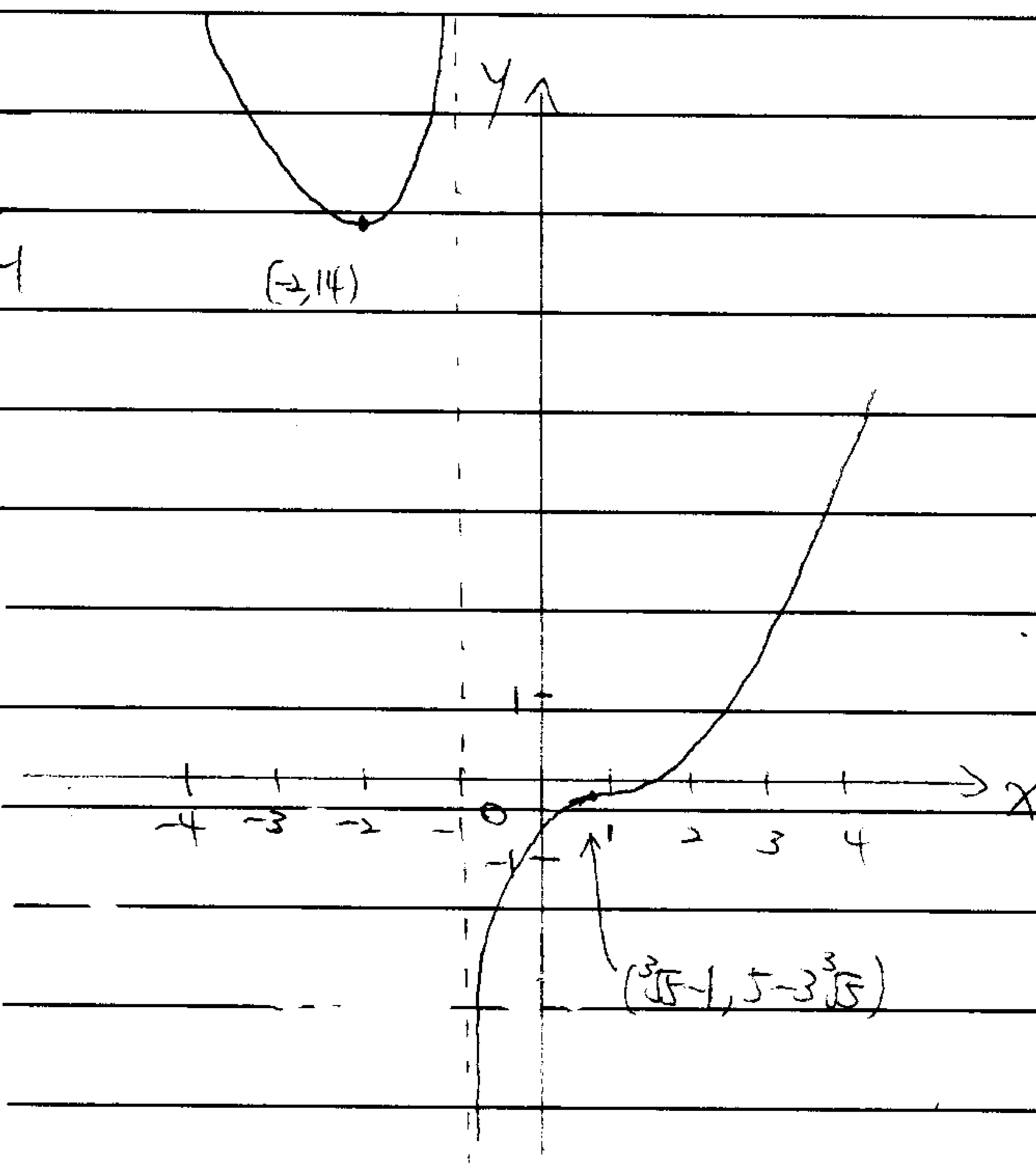
$\Rightarrow$ Critical points : $x=-1$ and $x=-2$

$\frac{d^2y}{dx^2}=0 \Rightarrow 2-\frac{10}{(x+1)^3}=0$ and $x\neq-1 \Rightarrow (x+1)^3=5 \Rightarrow x=\sqrt[3]{5}-1$

$\Rightarrow$ Inflection points : $x=-1$ and $x=\sqrt[3]{5}-1$

$\Rightarrow y(-2)=14$ and $y(\sqrt[3]{5}-1)=5-3\sqrt[3]{5}<0$

$\frac{dy}{dx}$	$-$	$+$	$+$	$+$
$\frac{d^2y}{dx^2}$	$+$	$+$	$-$	$+$
	$x<-2$	$-2<x<-1$	$-1<x<\sqrt[3]{5}-1$	$x>\sqrt[3]{5}-1$



2. Let $u = \sin x$, $\frac{du}{dx} = \cos x$

$$\Rightarrow \frac{d}{dx}\int_{\sin x}^{1} e^{t^2}\,dt = -\frac{d}{du}\int_{1}^{u} e^{t^2}\,dt \cdot \frac{du}{dx} = -e^{u^2}\cdot \cos x = -\cos x\, e^{\sin^2 x}$$

3. $$\lim_{n\to\infty}\sum_{k=\frac{n}{2}}^{n}\frac{n}{k^2} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=\frac{n}{2}}^{n}\frac{1}{(\frac{k}{n})^2} = \int_{\frac{1}{2}}^{1}\frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{\frac{1}{2}}^{1} = 1$$

4. Since $f(1) - f(0) = \int_0^1 f'(x)\,dx$

$$\Rightarrow f(1) = f(0) + \int_0^1 \sqrt{x^3+2}\,dx > -1 + \int_0^1 \sqrt{2}\,dx = \sqrt{2}-1 > 0$$

Moreover $f(0) = -1 < 0 \Rightarrow$ By Intermediate Value Theorem, $\exists\, c \in (0,1)$ such that $f(c) = 0$

Since $f'(x) = \sqrt{x^3+2} > 0$ on $0<x<1 \Rightarrow f$ is strictly increasing on $0<x<1$

Hence $f(x) = 0$ has exactly one solution on $(0,1)$

5. $\lim_{x\to0^+} \frac{e^{-(\frac{1}{x})}}{x} = \lim_{x\to0^+} \frac{\frac{1}{x}}{e^{\frac{1}{x}}} = \lim_{x\to0^+} \frac{\frac{-1}{x^2}}{\frac{-1}{x^2}e^{\frac{1}{x}}} = \lim_{x\to0^+} e^{-\frac{1}{x}} = 0$

6. (a) $y = 2^{(x^2)} \Rightarrow \frac{dy}{dx} = (\ln 2)\cdot 2^{(x^2)}\cdot(2x)$

(b) $y = \frac{(x+1)^{10}(x+2)^8}{(2x+1)^4} \Rightarrow \ln y = 10\ln(x+1) + 8\ln(x+2) - 4\ln(2x+1)$

$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \frac{10}{x+1} + \frac{8}{x+2} - \frac{8}{2x+1}$

$\Rightarrow \frac{dy}{dx} = \left(\frac{10}{x+1} + \frac{8}{x+2} - \frac{8}{2x+1}\right)\frac{(x+1)^{10}(x+2)^8}{(2x+1)^4}$

7. (a) $\int \cot x\,dx = \int \frac{\cos x}{\sin x}dx$, let $u = \sin x \Rightarrow du = \cos x\,dx$

$= \int \frac{1}{u}du = \ln|u| + C = \ln|\sin x| + C$, for some constant C

(b) $\int_{-1}^{1} x^2\sqrt{x^3+1}\,dx$, let $u = x^3+1 \Rightarrow du = 3x^2dx$

$= \frac{1}{3}\int_0^2 \sqrt{u}\,du = \frac{1}{3}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_0^2 = \frac{1}{3}\left(\frac{2}{3}\sqrt{8} - 0\right) = \frac{4}{9}\sqrt{2}$

8. $\frac{df^{-1}(y)}{dy}\Big|_{y=2} = \frac{1}{\frac{df(x)}{dx}\Big|_{x=1}}$ (Since $f(1)=2$)

$= \frac{1}{1.1}$

9. (a) let $y = x^{\ln(1-x)} \Rightarrow \ln y = \ln(x^{\ln(1-x)}) = \ln(1-x)\ln x$

$\Rightarrow \lim_{x\to0^+}\ln y = \lim_{x\to0^+}\frac{\ln(1-x)}{\frac{1}{\ln x}} = \lim_{x\to0^+}\frac{\frac{-1}{1-x}}{\frac{-1}{x(\ln x)^2}} = \lim_{x\to0^+}\frac{x(\ln x)^2}{1-x}$

Since $\lim_{x\to0^+} x(\ln x)^2 = \lim_{x\to0^+}\frac{(\ln x)^2}{\frac{1}{x}} = \lim_{x\to0^+}\frac{2(\ln x)\cdot\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to0^+} -2\frac{\ln x}{\frac{1}{x}} = \lim_{x\to0^+} -2\frac{\frac{1}{x}}{\frac{-1}{x^2}} = 0$

Hence $\lim_{x\to0^+}\ln y = 0 \Rightarrow \lim_{x\to0^+} x^{\ln(1-x)} = e^0 = 1$

9 (b) $\lim_{x\to 0} \frac{x^2 \sin\frac{1}{x}}{\sin x} = \lim_{x\to 0} \frac{x}{\sin x} \cdot x \cdot \sin\frac{1}{x} = 1 \cdot 0 = 0$

10. The Fundamental Theorem of Calculus, Part 1:

If f is continuous on $[a,b]$, then $F(x) = \int_a^x f(t)dt$ has a derivative at every point of $[a,b]$ and $\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)dt = f(x)$

The Fundamental Theorem of Calculus, Part 2:

If f is continuous on $[a,b]$ and F is any antiderivative of f on $[a,b]$, then $\int_a^b f(x)dx = F(b) - F(a)$

Proof of part 1 implies part 2:

Let F be any antiderivative of f on $[a,b]$.

By part 1, we have $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

$\Rightarrow F(x) = \int_a^x f(t)dt + C$, for some constant C

$\Rightarrow F(b) - F(a) = \left(\int_a^b f(t)dt + C\right) - \left(\int_a^a f(t)dt + C\right) = \int_a^b f(t)dt$