Midterm I

1.

$$\lim_{y\to\infty} y\sin\frac{2}{y} = \lim_{y\to\infty}\frac{\sin\frac{2}{y}}{\frac{1}{y}} = \lim_{y\to\infty} 2\,\frac{\sin\frac{2}{y}}{\frac{2}{y}} = 2$$

2.

(a) Given $\epsilon>0$, $\exists\,\delta>0$ s.t. if $0<|x-1|<\delta \Rightarrow |\sqrt{x}-1|<\epsilon$

(b) For $\epsilon=0.1$, we wish $|\sqrt{x}-1|<0.1 \iff -0.1<\sqrt{x}-1<0.1$

$$\iff 0.9<\sqrt{x}<1.1 \iff 0.81<x<1.21$$

$$\iff -0.19<x-1<0.21 \Rightarrow \text{pick } \delta=0.19$$

3. If $\lim_{x\to1} f(x)=3$ and $\lim_{x\to1} g(x)=4$, then

Given $\epsilon>0$, $\exists\,\delta_1>0$ s.t. if $0<|x-1|<\delta_1 \Rightarrow |f(x)-3|<\frac{\epsilon}{2}$

and $\exists\,\delta_2>0$ s.t. if $0<|x-1|<\delta_2 \Rightarrow |g(x)-4|<\frac{\epsilon}{2}$

$$\Rightarrow \exists\,\delta=\min\{\delta_1,\delta_2\} \text{ s.t. if } 0<|x-1|<\delta$$

$$\Rightarrow |f(x)+g(x)-7| = |f(x)-3+g(x)-4| \le |f(x)-3|+|g(x)-4| < \epsilon$$

Hence $\lim_{x\to1}(f(x)+g(x))=7$

4. (a) Intermediate Value Theorem:

If $f(x)$ is a continuous function on $[a,b]$,

then $\forall\, M$ between $f(a)$ & $f(b)$, $\exists\, c\in[a,b]$ such that $f(c)=M$.

4. (b) Since y is continuous on $\mathbb{R}$

and $y(0)=1>0$, $y(-1)=-2<0$

Hence, by I.V.T., there is a c between -1 & 0 s.t. $y(c)=0$

5. Let $f(x)=\sin x \Rightarrow \sin(\frac{\pi}{6}-0.01)=f(\frac{\pi}{6}-0.01)$

Let $L(x)$ be the linearization of $f(x)$ at $x=\frac{\pi}{6}$

$\Rightarrow L(x)=f(\frac{\pi}{6})+f'(\frac{\pi}{6})(x-\frac{\pi}{6})=\frac{1}{2}+\frac{\sqrt{3}}{2}(x-\frac{\pi}{6})$

$\Rightarrow f(\frac{\pi}{6}-0.01)\approx L(\frac{\pi}{6}-0.01)=\frac{1}{2}-\frac{0.01}{2}\sqrt{3}$ (approximate value)

Moreover, $\left|f(\frac{\pi}{6}-0.01)-L(\frac{\pi}{6}-0.01)\right|\le\frac{1}{2}\left(\max_{\frac{\pi}{6}-0.01\le\xi\le\frac{\pi}{6}}\left|-\sin\xi\right|\right)\left(\frac{\pi}{6}-0.01-\frac{\pi}{6}\right)^2$

$=\frac{1}{2}\cdot\sin\frac{\pi}{6}(-0.01)^2=2.5\times10^{-5}<5\times10^{-5}$

6. $y^4+(x-y)^2=1 \Rightarrow 4y^3y'+2(x-y)(1-y')=0 \quad \cdots(1)$

$\Rightarrow 4y'(1,1)+0=0 \Rightarrow y'(1,1)=0$

From (1) $\Rightarrow 12y^2(y')^2+4y^3y''+2(1-y')^2+2(x-y)(-y'')=0$

$\Rightarrow 12\cdot1\cdot(0)^2+4\cdot1\cdot y''(1,1)+2(1-0)^2+0=0$

$\Rightarrow y''(1,1)=-\frac{1}{2}$

7 (a) $y=(x+x^{\frac{1}{2}})^{-\frac{1}{2}} \Rightarrow \frac{dy}{dx}=-\frac{1}{2}(x+x^{\frac{1}{2}})^{-\frac{3}{2}}\cdot(1+\frac{1}{2}x^{-\frac{1}{2}})$

(b) $y=\tan(\sin(x^2+1)) \Rightarrow \frac{dy}{dx}=[\sec^2(\sin(x^2+1))]\cdot[\cos(x^2+1)]\cdot(2x)$

8. $f(x)=x^{\frac{1}{3}}(x-4)$

$\Rightarrow f'(x)=\frac{1}{3}x^{-\frac{2}{3}}(x-4)+x^{\frac{1}{3}}\cdot 1$

$=\frac{1}{3}x^{-\frac{2}{3}}(x-4+3x)=\frac{4}{3}x^{-\frac{2}{3}}(x-1)$

$\left.\begin{array}{l} x=0,\ f'(x) \text{ does not exist} \\ x=1,\ f'(x)=0 \end{array}\right\} \Rightarrow$ critical points : $x=0$ and $x=1$

$\Rightarrow f''(x)=\frac{4}{3}\left(-\frac{2}{3}x^{-\frac{5}{3}}(x-1)+x^{-\frac{2}{3}}\cdot 1\right)=\frac{4}{9}x^{-\frac{5}{3}}(-2(x-1)+3x)$

$=\frac{4}{9}x^{-\frac{5}{3}}(x+2)$

$\left.\begin{array}{l} x=0,\ f''(x) \text{ does not exist} \\ x=-2,\ f''(x)=0 \end{array}\right\} \Rightarrow$ Inflection points : $x=-2$ and $x=0$

	$x<-2$	$-2<x<0$	$0<x<1$	$x>1$
f'	$-$	$-$	$-$	$+$
f''	$+$	$-$	$+$	$+$

$f(-2)=\sqrt[3]{-2}(-2-4)=-6\sqrt[3]{-2}>6$, $f(1)=1\cdot(1-4)=-3$

$f(0)=0$