8.1-45

$$\int_{\frac{\pi}{2}}^{\pi} \sqrt{1+\cos 2t}\, dt = \int_{\frac{\pi}{2}}^{\pi} \sqrt{1+2\cos^2 t - 1}\, dt$$

$$= \int_{\frac{\pi}{2}}^{\pi} \sqrt{2\cos^2 t}\, dt = \sqrt{2}\int_{\frac{\pi}{2}}^{\pi} |\cos t|\, dt$$

$$= \sqrt{2}\int_{\frac{\pi}{2}}^{\pi} -\cos t\, dt = -\sqrt{2}\,[\sin t]_{\frac{\pi}{2}}^{\pi}$$

$$= \sqrt{2}$$

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$$\int \left((x^2-1)(x+1)\right)^{-\frac{2}{3}} dx \overset{\text{Let}}{=} A$$

(a)

Let $u = \frac{1}{x+1} \Rightarrow du = -\frac{1}{(x+1)^2}dx$ and $\frac{1}{1-2u} = \frac{x+1}{x-1}$

$$A = \int \left((x-1)(x+1)^2\right)^{-\frac{2}{3}} dx = \int (x-1)^{-\frac{2}{3}}(x+1)^{-\frac{4}{3}}\left[-(x+1)^2\right] du$$

$$= -\int (x-1)^{-\frac{2}{3}}(x+1)^{\frac{2}{3}} du = -\int \left(\frac{x+1}{x-1}\right)^{\frac{2}{3}} du$$

$$= -\int \left(\frac{1}{1-2u}\right)^{\frac{2}{3}} du = -\int (1-2u)^{-\frac{2}{3}} du$$

$$= \frac{3}{2}(1-2u)^{\frac{1}{3}} + C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}} + C$$

(b)

$u = \left(\frac{x-1}{x+1}\right)^k$, $k = 1, \frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, -\frac{2}{3}, -1$

(i) $k=1$

$u = \frac{x-1}{x+1} \Rightarrow du = \frac{2}{(x+1)^2}dx$

$$A = \int (x-1)^{-\frac{2}{3}}(x+1)^{-\frac{4}{3}}\left[\frac{1}{2}(x+1)^2\right] du$$

$$= \frac{1}{2}\int (x-1)^{-\frac{2}{3}}(x+1)^{\frac{2}{3}} du = \frac{1}{2}\int u^{-\frac{2}{3}} du = \frac{3}{2}u^{\frac{1}{3}} + C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}} + C$$

(ii) $k=\frac{1}{2}$

$u=\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}} \Rightarrow du=\frac{1}{2}\left(\frac{x-1}{x+1}\right)^{-\frac{1}{2}}\cdot\frac{2}{(x+1)^2}dx=\left(\frac{x-1}{x+1}\right)^{-\frac{1}{2}}\frac{1}{(x+1)^2}dx$

$$A=\int (x-1)^{-\frac{2}{3}}(x+1)^{-\frac{4}{3}}\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}}(x+1)^2\,du$$

$$=\int\left(\frac{x+1}{x-1}\right)^{\frac{2}{3}}\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}}du=\int\left(\frac{x-1}{x+1}\right)^{-\frac{1}{6}}du=\int u^{-\frac{1}{3}}du$$

$$=\frac{3}{2}u^{\frac{2}{3}}+C=\frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}}+C$$

(iii) $k=\frac{1}{3}$

$u=\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}} \Rightarrow du=\frac{1}{3}\left(\frac{x-1}{x+1}\right)^{-\frac{2}{3}}\cdot\frac{2}{(x+1)^2}dx=\frac{2}{3}\left(\frac{x-1}{x+1}\right)^{-\frac{2}{3}}\frac{1}{(x+1)^2}dx$

$$A=\int(x-1)^{-\frac{2}{3}}(x+1)^{-\frac{4}{3}}\cdot\frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{2}{3}}(x+1)^2\,du$$

$$=\frac{3}{2}\int\left(\frac{x+1}{x-1}\right)^{\frac{2}{3}}\left(\frac{x-1}{x+1}\right)^{\frac{2}{3}}du=\frac{3}{2}\int 1\,du=\frac{3}{2}u+C$$

$$=\frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}}+C$$

(iv) $k=-\frac{1}{3}$

$u=\left(\frac{x-1}{x+1}\right)^{-\frac{1}{3}} \Rightarrow du=-\frac{2}{3}\left(\frac{x-1}{x+1}\right)^{-\frac{4}{3}}\frac{1}{(x+1)^2}dx$

$$A=-\frac{3}{2}\int\left(\frac{x+1}{x-1}\right)^{\frac{2}{3}}\left(\frac{x-1}{x+1}\right)^{\frac{4}{3}}du=-\frac{3}{2}\int\left(\frac{x-1}{x+1}\right)^{\frac{2}{3}}du$$

$$=-\frac{3}{2}\int u^{-2}du=\frac{3}{2}u^{-1}+C=\frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}}+C$$

(v) $k=-\frac{2}{3}$

$u=\left(\frac{x-1}{x+1}\right)^{-\frac{2}{3}} \Rightarrow du=-\frac{4}{3}\left(\frac{x-1}{x+1}\right)^{-\frac{5}{3}}\frac{1}{(x+1)^2}dx$

$$A=-\frac{3}{4}\int\left(\frac{x+1}{x-1}\right)^{\frac{2}{3}}\left(\frac{x-1}{x+1}\right)^{\frac{5}{3}}du=-\frac{3}{4}\int\left(\frac{x-1}{x+1}\right)du=-\frac{3}{4}\int u^{-\frac{3}{2}}du$$

$$=-\frac{3}{4}\left(-2u^{-\frac{1}{2}}\right)+C=\frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}}+C$$

(vi) $k=-1$

$$u=\left(\frac{x-1}{x+1}\right)^{-1} \Rightarrow du=-2\left(\frac{x-1}{x+1}\right)^{-2}\frac{1}{(x+1)^2}dx$$

$$A=-\frac{1}{2}\int\left(\frac{x+1}{x-1}\right)^{\frac{2}{3}}\left(\frac{x-1}{x+1}\right)^{2}du=-\frac{1}{2}\int\left(\frac{x-1}{x+1}\right)^{\frac{4}{3}}du$$

$$=-\frac{1}{2}\int u^{-\frac{4}{3}}du=-\frac{1}{2}\left(-3u^{-\frac{1}{3}}\right)+C=\frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}}+C$$

(c)

$$u=\tan^{-1}x \Rightarrow \tan u=x \Rightarrow \sec^2 u\,du=dx$$

We will use these equalities

$$\begin{cases}\sin u+\cos u=\sin u+\sin\left(\frac{\pi}{2}-u\right)=2\sin\frac{\pi}{4}\cos\left(u-\frac{\pi}{4}\right)\\ \sin u-\cos u=\sin u-\sin\left(\frac{\pi}{2}-u\right)=2\cos\frac{\pi}{4}\sin\left(u-\frac{\pi}{4}\right)\end{cases}$$

$$A=\int(x+1)^{-2}\left(\frac{x-1}{x+1}\right)^{-\frac{2}{3}}dx=\int\frac{1}{(\tan u+1)^2}\left(\frac{\tan u-1}{\tan u+1}\right)^{-\frac{2}{3}}\frac{1}{\cos^2 u}du$$

$$=\int\frac{1}{(\sin u+\cos u)^2}\left(\frac{\sin u-\cos u}{\sin u+\cos u}\right)^{-\frac{2}{3}}du$$

$$=\int\frac{1}{2\cos^2\left(u-\frac{\pi}{4}\right)}\left[\frac{\sin\left(u-\frac{\pi}{4}\right)}{\cos\left(u-\frac{\pi}{4}\right)}\right]^{-\frac{2}{3}}du=\frac{1}{2}\int\tan^{-\frac{2}{3}}\left(u-\frac{\pi}{4}\right)\sec^2\left(u-\frac{\pi}{4}\right)du$$

$$=\frac{3}{2}\tan^{\frac{1}{3}}\left(u-\frac{\pi}{4}\right)+C=\frac{3}{2}\left[\frac{\tan u-\tan\frac{\pi}{4}}{1+\tan u\tan\frac{\pi}{4}}\right]^{\frac{1}{3}}+C=\frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}}+C$$

(d)

$$u=\tan^{-1}\sqrt{x} \Rightarrow \tan u=\sqrt{x} \Rightarrow \tan^2 u=x \Rightarrow 2\tan u\left(\sec^2 u\right)du=dx$$

$$\Rightarrow\begin{cases}x-1=\tan^2 u-1=\frac{\sin^2 u-\cos^2 u}{\cos^2 u}=\frac{1-2\cos^2 u}{\cos^2 u}\\ x+1=\tan^2 u+1=\sec^2 u=\frac{1}{\cos^2 u}\\ \cos 2u=\frac{1-\tan^2 u}{1+\tan^2 u}\end{cases}$$

$$A=\int(x-1)^{-\frac{2}{3}}(x+1)^{-\frac{4}{3}}dx=\int\frac{(1-2\cos^2 u)^{-\frac{2}{3}}}{(\cos^2 u)^{-\frac{2}{3}}}\cdot\frac{1}{(\cos^2 u)^{-\frac{4}{3}}}\cdot\frac{2\sin u}{\cos^3 u}du$$

$$= \int \frac{2\cos u}{(1-2\cos^2 u)^{\frac{4}{3}}} \sin u \, du \qquad \left(\begin{array}{l} \text{Let } w = \cos u \\ dw = -\sin u \, du \end{array} \right)$$

$$= -2 \int \frac{w}{(1-2w^2)^{\frac{4}{3}}} dw$$

$$= -2\left(\frac{3}{4}(1-2w^2)^{-\frac{1}{3}}\right) + C = \frac{3}{2}(1-2\cos^2 u)^{-\frac{1}{3}} + C$$

$$= \frac{3}{2}(-\cos 2u)^{-\frac{1}{3}} + C = \frac{3}{2}\left(-\frac{1-\tan^2 u}{1+\tan^2 u}\right)^{-\frac{1}{3}} + C$$

$$= \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}} + C$$

(e) $u = \tan^{-1}\left(\frac{x-1}{2}\right) \Rightarrow \tan u = \frac{x-1}{2} \Rightarrow x+1 = 2(\tan u + 1)$

$\Rightarrow 2\sec^2 u \, du = dx$

$$A = \int (x-1)^{-\frac{2}{3}}(x+1)^{-\frac{4}{3}} dx = \int (2\tan u)^{-\frac{2}{3}}(2(\tan u+1))^{-\frac{4}{3}} 2\sec^2 u \, du$$

$$= \frac{1}{2}\int \left(\frac{\tan u}{1+\tan u}\right)^{-\frac{2}{3}} \frac{1}{(1+\tan u)^2} \sec^2 u \, du$$

$$= \frac{1}{2}\int \left(1-\frac{1}{1+\tan u}\right)^{-\frac{2}{3}} \frac{\sec^2 u}{(1+\tan u)^2} du \qquad \left(\begin{array}{l} \text{let } w = 1 - \frac{1}{1+\tan u} \\ dw = \frac{\sec^2 u}{(1+\tan u)^2} du \end{array} \right)$$

$$= \frac{1}{2}\int w^{-\frac{2}{3}} dw$$

$$= \frac{3}{2} w^{\frac{1}{3}} + C = \frac{3}{2}\left(1 - \frac{1}{\tan u + 1}\right)^{\frac{1}{3}} + C = \frac{3}{2}\left(1 - \frac{2}{x+1}\right)^{\frac{1}{3}} + C$$

$$= \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}} + C$$

(f) $u = \cos^{-1} x \Rightarrow \cos u = x \Rightarrow -\sin u \, du = dx$

$$A = \int \left((\cos^2 u - 1)(\cos u + 1)\right)^{-\frac{2}{3}} (-\sin u) \, du$$

$$= -\int \frac{\sin u}{\sqrt[3]{(-\sin^2 u)^2 (2\cos^2 \frac{u}{2})^2}} du = -\int \frac{1}{\sqrt[3]{(\sin u)\, 2^2 \cdot \cos^4 \frac{u}{2}}} du$$

$$= -\int \frac{1}{\sqrt[3]{2^3 \sin\frac{u}{2} \cos^5\frac{u}{2}}}\,du = -\frac{1}{2}\int \frac{1}{\left(\frac{\sin\frac{u}{2}}{\cos\frac{u}{2}}\right)^{\frac{1}{3}}\left(\cos^6\frac{u}{2}\right)^{\frac{1}{3}}}\,du$$

$$= -\frac{1}{2}\int \tan\left(\tfrac{u}{2}\right)^{-\frac{1}{3}} \cdot \sec^2\left(\tfrac{u}{2}\right) du \qquad \left(\begin{array}{l}\text{let } w = \tan\left(\frac{u}{2}\right) \\ dw = \left[\sec^2\left(\frac{u}{2}\right)\right]\frac{1}{2}\,du\end{array}\right)$$

$$= -\frac{1}{2}\int 2 w^{-\frac{1}{3}}\,dw$$

$$= -1\left(\tfrac{3}{2} w^{\frac{2}{3}}\right) + C = -\frac{3}{2}\left[\tan\left(\tfrac{u}{2}\right)\right]^{\frac{2}{3}} + C$$

$$= -\frac{3}{2}\left(\tan^2\left(\tfrac{u}{2}\right)\right)^{\frac{1}{3}} + C = -\frac{3}{2}\left(\frac{1-\cos u}{1+\cos u}\right)^{\frac{1}{3}} + C$$

$$= \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}} + C$$

(g) $u = \cosh^{-1}x \Rightarrow \cosh u = x \Rightarrow \sinh u\,du = dx$

$$A = \int \left((\cosh^2 u - 1)(\cosh u + 1)\right)^{-\frac{2}{3}} \sinh u\,du$$

$$= \int \left[(\sinh^2 u)^2 \left(2\cosh^2\left(\tfrac{u}{2}\right)\right)^2\right]^{-\frac{1}{3}} \sinh u\,du$$

$$= \int \left[(\sinh u)\left(2^2\cosh^4\left(\tfrac{u}{2}\right)\right)\right]^{-\frac{1}{3}} du = \int \left[\left(2\sinh\left(\tfrac{u}{2}\right)\cosh\left(\tfrac{u}{2}\right)\right)\left(2^2\cosh^4\left(\tfrac{u}{2}\right)\right)\right]^{-\frac{1}{3}} du$$

$$= \frac{1}{2}\int \left(\frac{\sinh\left(\frac{u}{2}\right)}{\cosh\left(\frac{u}{2}\right)} \cdot \cosh^6\left(\tfrac{u}{2}\right)\right)^{-\frac{1}{3}} du$$

$$= \frac{1}{2}\int \tanh^{-\frac{1}{3}}\left(\tfrac{u}{2}\right) \cdot \operatorname{sech}^2\left(\tfrac{u}{2}\right) du \qquad \left(\begin{array}{l}\text{let } w = \tanh\left(\frac{u}{2}\right) \\ dw = \left[\operatorname{sech}^2\left(\frac{u}{2}\right)\right]\frac{1}{2}\,du\end{array}\right)$$

$$= \frac{1}{2}\int 2 w^{-\frac{1}{3}}\,dw$$

$$= \frac{3}{2} w^{\frac{2}{3}} + C = \frac{3}{2}\left(\tanh^2\left(\tfrac{u}{2}\right)\right)^{\frac{1}{3}} + C = \frac{3}{2}\left(\frac{\cosh u - 1}{\cosh u + 1}\right)^{\frac{1}{3}} + C$$

$$= \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}} + C$$

Problem 2

$\int x\sqrt{x+1}\,dx$, $x>-1$ let $x+1=y^2$

$\Rightarrow dx=2y\,dy$

$=\int (y^2-1)y\cdot 2y\,dy$

$=\int 2y^4-2y^2\,dy=\frac{2}{5}y^5-\frac{2}{3}y^3+C$

$=\frac{2}{5}(\sqrt{x+1})^5-\frac{2}{3}(\sqrt{x+1})^3+C$

8.2-25

$$\int e^{\sqrt{3s+9}}\,ds=\int\sqrt{3s+9}\cdot\frac{e^{\sqrt{3s+9}}}{\sqrt{3s+9}}\,ds \quad \left(\begin{aligned}&\text{let } u=\sqrt{3s+9}\\ &dv=\frac{e^{\sqrt{3s+9}}}{\sqrt{3s+9}}\,ds\end{aligned}\right)$$

$$=\frac{2}{3}\sqrt{3s+9}\,e^{\sqrt{3s+9}}-\int\frac{2}{3}e^{\sqrt{3s+9}}\cdot\frac{3}{2\sqrt{3s+9}}\,ds$$

$$=\frac{2}{3}\sqrt{3s+9}\,e^{\sqrt{3s+9}}-\frac{2}{3}e^{\sqrt{3s+9}}+C=\frac{2}{3}\left(\sqrt{3s+9}\,e^{\sqrt{3s+9}}-e^{\sqrt{3s+9}}\right)+C$$

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$$\int z(\ln z)^2\,dz \quad \left(\begin{aligned}&\text{let } u=(\ln z)^2\\ &dv=z\,dz\end{aligned}\right)$$

$$=\frac{z^2}{2}(\ln z)^2-\int\frac{z^2}{2}\left(2\ln z\cdot\frac{1}{z}\right)dz=\frac{z^2}{2}(\ln z)^2-\int z\ln z\,dz$$

$$=\frac{z^2}{2}(\ln z)^2-\left(\frac{z^2}{2}\ln z-\int\frac{z^2}{2}\cdot\frac{1}{z}\,dz\right)=\frac{z^2}{2}(\ln z)^2-\frac{z^2}{2}\ln z+\frac{z^2}{4}+C$$

8.3-42

$$\int \frac{1}{\sqrt{\theta}+\sqrt[3]{\theta}}\,d\theta \qquad \left(\text{let } \theta = x^6 \Rightarrow d\theta = 6x^5dx\right)$$

$$= \int \frac{6x^5}{x^3+x^2}\,dx$$

$$= \int \frac{6x^3}{x+1}\,dx = \int 6x^2-6x+6\,dx - 6\int \frac{1}{x+1}\,dx$$

$$= 2x^3-3x^2+6x-6\ln|x+1|+C$$

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$$\int t\ln(t+5)\,dt \qquad \left(\begin{array}{ll} u=\ln(t+5), & dv = t\,dt \\ \Rightarrow du = \frac{1}{t+5}dt, & v=\frac{t^2}{2} \end{array}\right)$$

$$= \frac{t^2}{2}\ln(t+5) - \int \frac{t^2}{2}\frac{1}{t+5}\,dt$$

$$= \frac{t^2}{2}\ln(t+5) - \frac{1}{2}\int \frac{t^2}{t+5}\,dt = \frac{t^2}{2}\ln(t+5) - \frac{1}{2}\left(\int t-5\,dt + \int \frac{25}{t+5}\,dt\right)$$

$$= \frac{t^2}{2}\ln(t+5) - \frac{1}{2}\left(\frac{t^2}{2}-5t+25\ln|t+5|\right)+C$$

$$= \frac{t^2}{2}\ln(t+5) - \frac{t^2}{4}+\frac{5}{2}t-\frac{25}{2}\ln|t+5|+C$$