Final

1. (a) $\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0 \Rightarrow \ln x = o(x^{-1})$ as $x \to 0^+$

$\Rightarrow$ It's true.

(b) $\lim_{x\to 0^+} \frac{\sqrt{x}}{1} = 0 \Rightarrow \sqrt{x} = o(1)$ as $x \to 0^+$

and $x = O(x)$

But $\lim_{x\to 0^+} \frac{\sqrt{x}+x}{x} = \lim_{x\to 0^+} \frac{1+\sqrt{x}}{\sqrt{x}} = \infty \Rightarrow o(1) + O(x) = O(x)$ as $x \to 0^+$ is false.

2. $\text{Volume} = \int_{-1}^{1} \pi \left[(2+\sqrt{1-y^2})^2 - (2-\sqrt{1-y^2})^2 \right] dy$

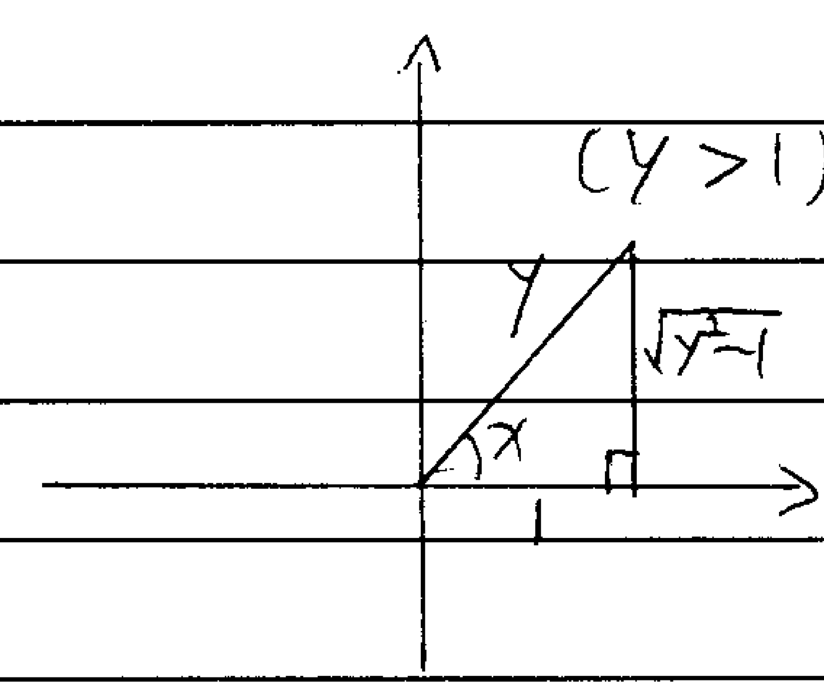
$= 8\pi \int_{-1}^{1} \sqrt{1-y^2}\, dy = 8\pi \cdot \frac{\pi}{2} = 4\pi^2$

$\text{Surface} = \int_{-1}^{1} 2\pi (2+\sqrt{1-y^2}) \sqrt{1+\frac{y^2}{1-y^2}}\, dy + \int_{-1}^{1} 2\pi (2-\sqrt{1-y^2}) \sqrt{1+\frac{y^2}{1-y^2}}\, dy$

$= 8\pi \int_{-1}^{1} \frac{1}{\sqrt{1-y^2}} dy = 8\pi \cdot \sin^{-1} y \Big|_{-1}^{1} = 8\pi \cdot \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 8\pi^2$

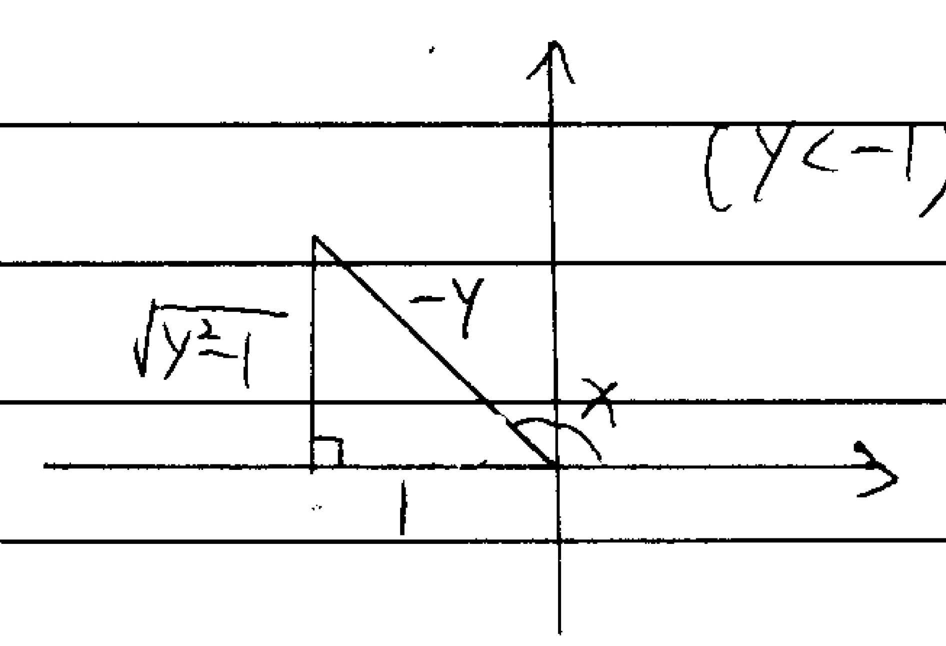
3. Let $\sec^{-1} y = x \Rightarrow \sec x = y \Rightarrow \sec x \cdot \tan x \cdot \frac{dx}{dy} = 1$

$\Rightarrow \frac{dx}{dy} = \frac{1}{\sec x \cdot \tan x}$



$y > 1 \Rightarrow \sec x = y$ and $\tan x = \sqrt{y^2-1}$

$y < -1 \Rightarrow \sec x = y$ and $\tan x = -\sqrt{y^2-1}$



$\Rightarrow \frac{d \sec^{-1} y}{dy} = \begin{cases} \frac{1}{y\sqrt{y^2-1}}, & \text{if } y > 1 \\ \frac{-1}{y\sqrt{y^2-1}}, & \text{if } y < -1 \end{cases}$

$\Rightarrow \frac{d \sec^{-1} y}{dy} = \frac{1}{|y|\sqrt{y^2-1}}, \quad |y| > 1$

4. $\int_0^{\frac{\pi}{4}} \sin^3 x \cos^2 x\, dx$, let $\begin{cases} u = \cos x \\ du = -\sin x\, dx \end{cases}$

$= \int_1^{\frac{\sqrt{2}}{2}} -(1-u^2)\, u^2\, du = \int_1^{\frac{\sqrt{2}}{2}} u^4 - u^2\, du$

$= \frac{u^5}{5} - \frac{u^3}{3} \Big|_1^{\frac{\sqrt{2}}{2}} = \frac{2}{15} - \frac{7}{120}\sqrt{2}$

5. $\int_1^{e^\pi} \sin(\ln x)\, dx = x \sin(\ln x) \Big|_1^{e^\pi} - \int_1^{e^\pi} \cos(\ln x)\, dx$

$= -x\cos(\ln x) \Big|_1^{e^\pi} - \int_1^{e^\pi} \sin(\ln x)\, dx$

$\Rightarrow \int_1^{e^\pi} \sin(\ln x)\, dx = \frac{1}{2}\left[-e^\pi \cdot (-1) - (-1)\cdot 1\right] = \frac{e^\pi + 1}{2}$

6. $\int_0^{\frac{\pi}{2}} \frac{1}{2+\sin x}\, dx$, let $z = \tan\frac{x}{2} \Rightarrow \sin x = \frac{2z}{1+z^2}$

$dx = \frac{2}{1+z^2}\, dz$

$= \int_0^1 \frac{\frac{2}{1+z^2}}{2 + \frac{2z}{1+z^2}}\, dz = \int_0^1 \frac{1}{1+z^2+z}\, dz$

$= \int_0^1 \frac{1}{(z+\frac{1}{2})^2 + \frac{3}{4}}\, dz = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2}{\sqrt{3}}(z+\frac{1}{2})\right) \Big|_0^1 = \frac{2}{\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\pi}{3\sqrt{3}}$

7. $\int_1^2 \coth x \operatorname{csch} x\, dx = -\operatorname{csch} x \Big|_1^2 = \operatorname{csch} 1 - \operatorname{csch} 2$

8. $\int \frac{x^2}{\sqrt{4x - x^2}}\, dx = \int \frac{x^2}{\sqrt{4-(x-2)^2}}\, dx$, let $x-2 = 2\sin\theta \Rightarrow dx = 2\cos\theta\, d\theta$

$= \int \frac{(2\sin\theta + 2)^2}{2\cos\theta} \cdot 2\cos\theta\, d\theta = \int 4\sin^2\theta + 8\sin\theta + 4\, d\theta$

$= \int 4 \cdot \frac{1-\cos 2\theta}{2}\, d\theta - 8\cos\theta + 4\theta$

$= 2\theta - \sin 2\theta - 8\cos\theta + 4\theta + C$

$= 6\theta - \sin 2\theta - 8\cos\theta + C$

$= 6\sin^{-1}\left(\frac{x-2}{2}\right) - (x-2)\sqrt{1-\left(\frac{x-2}{2}\right)^2} - 8\sqrt{1-\left(\frac{x-2}{2}\right)^2} + C$

9. $\int \frac{x^3}{(x^2+1)^2}dx = \int \frac{x}{x^2+1}dx + \int \frac{-x}{(x^2+1)^2}dx$ (let $y = x^2+1 \Rightarrow dy = 2x\,dx$)

$= \frac{1}{2}\int \frac{1}{y}dy - \frac{1}{2}\int \frac{1}{y^2}dy$

$= \frac{1}{2}\ln|x^2+1| + \frac{1}{2}\frac{1}{(x^2+1)} + C$

10. $\int_1^\infty \frac{1}{\sqrt{x^3-x}}dx = \int_1^2 \frac{1}{\sqrt{x^3-x}}dx + \int_2^\infty \frac{1}{\sqrt{x^3-x}}dx$

(i) Consider $\int_1^2 \frac{1}{\sqrt{x^3-x}}dx$

Since $0 \le \sqrt{x^2-1} \le \sqrt{x}\sqrt{x^2-1} = \sqrt{x^3-x}$ when $1 \le x \le 2$

$\Rightarrow \frac{1}{\sqrt{x^3-x}} \le \frac{1}{\sqrt{x^2-1}}$ when $1 \le x \le 2$

Moreover, $\int_1^2 \frac{1}{\sqrt{x^2-1}}dx = \lim_{b\to 1^+}\int_b^2 \frac{1}{\sqrt{x^2-1}}dx = \lim_{b\to 1^+} \cosh^{-1}x\Big|_b^2$

$= \lim_{b\to 1^+} \cosh^{-1}2 - \cosh^{-1}b = \cosh^{-1}2$

$\Rightarrow \int_1^2 \frac{1}{\sqrt{x^2-1}}dx$ converges $\Rightarrow \int_1^2 \frac{1}{\sqrt{x^3-x}}dx$ converges

(ii) Consider $\int_2^\infty \frac{1}{\sqrt{x^3-x}}dx$

Since $\lim_{x\to\infty} \frac{\frac{1}{\sqrt{x^3-x}}}{\frac{1}{\sqrt{x^3}}} = \lim_{x\to\infty} \frac{\sqrt{x^3}}{\sqrt{x^3-x}} = \lim_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{x^2}}} = 1$

and $\int_2^\infty \frac{1}{\sqrt{x^3}}dx = \lim_{a\to\infty}\int_2^a x^{-\frac{3}{2}}dx = \lim_{a\to\infty} -2x^{-\frac{1}{2}}\Big|_2^a = \lim_{a\to\infty} \sqrt{2} - \frac{2}{\sqrt{a}} = \sqrt{2}$

$\Rightarrow \int_2^\infty \frac{1}{\sqrt{x^3}}dx$ converges $\Rightarrow \int_2^\infty \frac{1}{\sqrt{x^3-x}}dx$ converges

By (i), (ii) $\Rightarrow \int_1^\infty \frac{1}{\sqrt{x^3-x}}dx = \int_1^2 \frac{1}{\sqrt{x^3-x}}dx + \int_2^\infty \frac{1}{\sqrt{x^3-x}}dx$ converges