Homework Assignment for Chap 09

1. Section 9.1: Problems: 7, 41, 57, 59, 60, 65, 71, 72(abe).
2. Section 9.2: Problems: 14, 20, 33, 62, 63.
3. Section 9.3: Problems: 23, 24, 27, 39, 41, 47, 61, 62.
4. Section 9.4: Problems 3, 5, 7, 9, 11, 13, 15, 19, 23, 25, 28, 34, 35, 39, 40, 41, 44 (Does it converge? Why?).
5. Section 9.5: Problems 4, 8, 7, 9, 24, 25, 33, 35, 37, 43, 45, 47.
6. We know that if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. How about the counter part for integration?

   Suppose that $f(x)$ is non-negative, continuous on $[0,\infty)$ and $\int_0^{\infty} f(x)\,dx < \infty$. Is $\lim_{x\to\infty} f(x) = 0$ necessarily true?
7. Is $\sum_{n=1}^{\infty} \sin\frac{1}{n}$ convergent? How about $\sum_{n=1}^{\infty}(1-\cos\frac{1}{n})$?
8. We know that the $p = 1/2$ series $\sum_{k=1}^{\infty} k^{\frac{-1}{2}}$ diverges. The question here is how fast does the partial sum grows with n, or $\sum_{k=1}^{n} k^{\frac{-1}{2}} = O(n^?)$? In other words, can you evaluate

   $$\lim_{n\to\infty} \frac{\log\left(\sum_{k=1}^{n} k^{\frac{-1}{2}}\right)}{\log n}$$ if it exists?
9. Section 9.6: Problems 11, 15, 21, 23, 27, 28, 35, 39, 41, 44, 47.
10. Find a power series whose interval of convergence is $[1,3)$. Do the same for $(1,3)$, $[1,3]$ and $(1,3]$, respectively.
11. Section 9.7: Problems 1, 3, 7, 15, 19, 25, 29, 33 (show that equality holds), 35, 47(a), 50, 57, 58.
12. Continue on problem 50. Show that $f'(0) = 0$ and $f''(0) = 0$.

    Remark: In fact, it can be shown that $f^{(n)}(0) = 0$ for all n by induction. The calculation is lengthier, but not more difficult.

13. Prove the following version of the Taylor's Theorem.

    (a) If $f, f', \cdots, f^{(n+1)}$ are all continuous in $(a-h, a+h)$, $h > 0$. Then for any $x \in (a-h, a+h)$, we have

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \frac{f^{(n)}(a)}{n!}(x-a)^2 + R_n(x),$$

    where

$$R_n(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt. \quad (1)$$

    (b) Show that (1) leads to

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad (2)$$

    for some c between a and x.

14. Find the interval of convergence for the power series

$$1 + \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \cdots + \frac{1\cdots(2n-1)}{2\cdots 2n}x^n + \cdots.$$

    Hint: Denote by $a_n = \frac{1\cdots(2n-1)}{2\cdots 2n}$ and define $b_2 = \frac{2}{3}$, $b_3 = \frac{2\cdot 4}{3\cdot 5}, \cdots, b_n = \frac{2\cdots(2n-2)}{3\cdots(2n-1)}$.

    For $x = 1$, the fact that $b_n < 1$ will help.

    For $x = -1$, compare a_n with b_n and $2b_n$ to conclude that $a_n \sim n^{-p}$. What is p?

    Do the same for

$$1 + \frac{1}{3}x + \frac{1\cdot 4}{3\cdot 6}x^2 + \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}x^3 + \cdots + \frac{1\cdots(3n-2)}{3\cdots 3n}x^n + \cdots.$$

15. Section 9.8: Problems 9, 13, 28, 29, 31, 32, 33, 34(a), 35, 43.

16. Chap 9: Problems 27, 28, 29, 30, 32, 37, 47(a), 49-54(important), 71, 72.

17. Show that the Taylor series generated by $(1+x)^m$, $m \in R$, converges on $|x| < 1$. This does not mean that the series indeed converges to $(1+x)^m$. Analyzing $R_n(x)$ directly is not quite easy (try it and youll see why). An alternative approach is through the following steps:

    (a) Verify that

$$(k+1)\binom{m}{k+1} + k\binom{m}{k} = m\binom{m}{k}$$

(b) Define, for $|x| < 1$,

$$f(x) = \sum_{k=0}^{\infty} \binom{m}{k} x^k = 1 + mx + \frac{m(m-1)}{2!} x^2 \frac{m(m-1)(m-2)}{3!} x^3 + \cdots$$

Show that $f(0) = 1$ and

$$(1+x)f'(x) = mf(x), \quad |x| < 1.$$

(c) Show that $f(x) = (1+x)^m$ on $|x| < 1$.

18. Solve the differential equation

$$dy/dx = 1 + y^2, \quad y(0) = 0,$$

by power series expansion. That is, assume $y(x) = a_0 + a_1 x + a_2 x^2 + \cdots$, then compare the coefficients on both sides to solve for $a_0, a_1, \cdots$, successively. The differential equation can also be integrated directly. Verify your answer by comparing the first 2 nonzero coefficients of the Taylor series expansion of the exact solution.