Homework Assignment for Week 04

1. Section 9.7: Problems 1, 3, 7, 15, 19, 25, 29, 33 (show that equality holds), 35, 47(a), 50, 57, 58.

2. (s9.7-extra1) Continue on problem 50. Show that $f'(0) = 0$ and $f''(0) = 0$.

   Remark: In fact, it can be shown that $f^{(n)}(0) = 0$ for all n by induction. The calculation is lengthier, but not more difficult.

3. (s9.7-extra2) Prove the following version of the Taylor's Theorem.

   (a) If $f, f', \cdots, f^{(n+1)}$ are all continuous in $(a-h, a+h)$, $h > 0$. Then for any $x \in (a-h, a+h)$, we have

   $$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \frac{f^{(n)}(a)}{n!}(x-a)^2 + R_n(x),$$

   where

   $$R_n(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt. \tag{1}$$

   (b) Show that (1) leads to

   $$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \tag{2}$$

   for some c between a and x.

4. Read Taylor's Theorem and proof on p609-610. Note the difference on the assumption on $f^{(n+1)}$ between the two versions. The version in problem 3 has stronger assumption and leads to stronger result (1). On the other hand, to get (2), the weaker assumption in Theorem 15 on p609 is enough.

5. (s9.8-extra1) Show that the Taylor series generated by $(1+x)^m$, $m \in R$, converges on $|x| < 1$. This does not mean that the series indeed converges to $(1+x)^m$. Analyzing $R_n(x)$ directly is not quite easy (try it and youll see why). An alternative approach is through the following steps:

   (a) Verify that

   $$(k+1)\binom{m}{k+1} + k\binom{m}{k} = m\binom{m}{k}$$

(b) Define, for $|x| < 1$,

$$f(x) = \sum_{k=0}^{\infty} \binom{m}{k} x^k = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots$$

Show that $f(0) = 1$ and

$$(1+x)f'(x) = mf(x), \quad |x| < 1.$$

(c) Show that $f(x) = (1+x)^m$ on $|x| < 1$.