Midterm Exam 1

Mar 28, 2013, 10:10 AM. Show all details.

1. (16 pts) Is the integral $\int_0^\infty \frac{1}{\sqrt[3]{x+x^2}}dx$ convergent? Find all possible locations of divergence and check out one by one. Do not try to find the anti-derivative.

2. (10+6 pts) True or False? Prove it if true, give a counter example if false.

   (a) If $a_0+a_1x+a_2x^2+\cdots$ converges at $x=-2$, then it converges absolutely at $x=1$.

   (b) If $a_0+a_1x+a_2x^2+\cdots$ converges at $x=-1$, then it converges at $x=1$.

3. (16 pts) Evaluate $\lim_{n\to\infty} \frac{\log\left(\sum_{k=1}^n k^{\frac{-1}{2}}\right)}{\log n}$. Give details.

   Hint: If the limit is p, this means that the sum is approximately n^p. Find p. To prove it, recall the proof of integral test.

4. (6+10 pts)

   (a) Show that the series $1-\frac{\pi^2}{4\cdot 2!}+\frac{\pi^4}{16\cdot 4!}-\cdots+(-1)^n\frac{\pi^{2n}}{2^{2n}\cdot(2n)!}+\cdots$ converges *absolutely*. Explain.

   (b) Find the sum of the series in (a). Prove your answer.
   Hint: it is related to the Taylor series of an elementary function.

5. (10 pts) Suppose that $y(x)$ is given by

$$\frac{dy}{dx}=y+\cos(y-1), \qquad y(0)=1$$

   and $y(x)$ admits a Taylor series expansion around $x=0$. Find a_0, a_1, a_2 and a_3 for the expansion of $y(x)$.

6. (10+10 pts) True or False? Prove it if true, give a counter example if false.

   (a) If $f(x)=a_0+a_1x+a_2x^2+\cdots$ on $|x|<1$, then $a_n=f^{(n)}(0)/n!$.

   (b) If $g(x)=f(0)+\sum_{n=1}^\infty \frac{f^{(n)}(0)}{n!}x^n$ on $|x|<1$, then $f(x)=g(x)$ on $|x|<1$.

7. (10 pts) Give an approximation of $\int_0^{\frac{1}{2}}\sin(x^2)dx$ to within 10^{-8}. Give the formula of the approximation, need not give the numerical value. Explain why the error is less than 10^{-8}.

8. (10 pts) Let a parametrized curve be given by $r=e^{2\theta}$ in polar coordinate. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ of the curve at $\theta=\pi/2$.