Suggested Reading 2.1-2.3

Informal Definition of Limit (of function's value)

If $f(x) \longrightarrow L$ when $x \to c$

we say $\lim_{x \to c} f(x) = L$

Remark: The definition of $\lim_{x \to c} f(x)$ is not related to "$f(c)$".

Examples: (1) $f(x) = x$

(2) $f(x) = k$ (constant function)

(3) $f(x) =$ polynomials and rational functions

Properties of limits

If $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ both exists (and finite)

Then

- $\lim_{x \to c} \left(f(x) \pm g(x) \right) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$
- $\lim_{x \to c} \left(f(x) \cdot g(x) \right) = \left(\lim_{x \to c} f(x) \right) \cdot \left(\lim_{x \to c} g(x) \right)$
- $\lim_{x \to c} \left(k f(x) \right) = k \lim_{x \to c} f(x)$
- $\lim_{x \to c} \left(\dfrac{f(x)}{g(x)} \right) = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$, Provided $\lim_{x \to c} g(x) \neq 0$

Corollary: (1), (2) $\Rightarrow$ (3)

Examples of "$\lim_{x \to c} f(x)$ does not exist"

(1) $f(x) = [x]$ = integer part of x

i.e. $f(x) = n$ iff $n \leq x < n+1$

↑ (if and only if)

integer

(2) ~~$f(x) = \sin\left(\frac{1}{x}\right)$~~

$f(x) = \sin\left(\frac{1}{x}\right)$, $x \neq 0$ ($f(0)$ is not relevant)

In (1): we have $\lim_{x \to n^-} f(x) = n-1$

while $\lim_{x \to n^+} f(x) = n$

$\lim_{x \to n^-} \neq \lim_{x \to n^+}$ so $\lim_{x \to n}$ does not exist.

In (2) $f\left(\frac{1}{n\pi}\right) = 0$ $n = 1, 2, 3, \ldots$

while $f\left(\frac{1}{(2n \pm \frac{1}{2})\pi}\right) = \pm 1$,

values of $f(x)$ oscillates between $[-1, 1]$

as $x \to 0$.

Example (3):

$\lim_{x \to 0} \frac{1}{x^2} = ?$

We call it ∞ instead of non-existent.

But $\lim_{x \to 0} \frac{1}{x}$ does not exist; (Why?)

Limits involving Infinity

"$x \to \infty$" means x gets larger and larger

$x \to -\infty$ means x gets smaller and smaller
(@ taking account the sign)
(i.e. ~~large on and negative~~)
(i.e. negative with large absolute value)

Examples:

- $\lim_{x \to \infty} \frac{1}{x} = 0$ ("0^+", to be more specific)
- $\lim_{x \to 0^+} \frac{1}{x} = \infty$
- $\lim_{x \to 0^-} \frac{1}{x} = -\infty$
- $\lim_{x \to -\infty} \frac{1}{x} = 0^-$
- Properties of limits on Page 2-1 remains valid if $x \to c$ is replaced by $x \to \infty$ or $x \to -\infty$

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Limits of rational functions as $x \to \pm\infty$

$$\lim_{x \to a} \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0} = \frac{\pm\infty}{\pm\infty} = ?$$

Trick: Find $p = \max(m, n)$ and devide by x^p on ~~top and~~ numerator and denominator

Conclusion:

- $n < m$, $\lim_{x \to \pm\infty} \frac{(\ )}{(\ )} = 0$
- $n = m$, $\lim_{x \to \pm\infty} \frac{(\ )}{(\ )} = \frac{a_n}{b_n}$
- $n > m$, $\lim_{x \to \pm\infty} \frac{(\ )}{(\ )} = \infty$ or $-\infty$

Definition of "$\lim_{x \to c} f(x) = L$" Using ε, δ

How to specify "as $x \to c$" and "$f(x) \to L$" ?

and "$x \to c$" implies "$f(x) \to L$"

or "$f(x) \to L$ as $x \to c$" ?

Ans: Definition of $\lim_{x \to c} f(x) = L$

~~Ans~~: Given $\varepsilon > 0$, we can always find a $\delta > 0$ such that

"$0 < |x - c| < \delta$ implies $|f(x) - L| < \varepsilon$"

Remark: (1) This definition is "operational", to verify whether $\lim_{x \to c} f(x) = L$, one needs to develop a strategy to find a $\delta > 0$ for any given $\varepsilon > 0$

(2) We do not care whether $f(c) = L$, therefore we use "$0 < |x - c| < \delta$" instead of "$|x - c| < \delta$"

, If we use "$|x - c| < \delta$", the definition will impose "$\lim_{x \to c} f(x) = L$" AND "$f(c) = L$"

Example : Verify $\lim_{x\to 6} x^2 = 6^2$

Step 1. Given any $\varepsilon>0$, find an interval around 6 on which $|x^2-6^2|<\varepsilon$, that is $\sqrt{6^2-\varepsilon}<x<\sqrt{6^2+\varepsilon}$

(we may assume ε small so that $6^2-\varepsilon>0$)

Step 2 : Try to fit @ an interval of the form

$(6-\delta, 6)\cup(6, 6+\delta)$ { that is, $0<|x-6|<\delta$ }

inside $(\sqrt{6^2-\varepsilon}, \sqrt{6^2+\varepsilon})$

For this purpose, it suffices to take

$\delta = \min\{6-\sqrt{6^2-\varepsilon}, \sqrt{6^2+\varepsilon}-6\}$

(or any smaller positive number)

Step 3 Check indeed that

$0<|x-6|<\delta$ implies $|x^2-6^2|<\varepsilon$

Detail left as exercise.

Homework : (1) Try to define $\lim_{x\to c^+} f(x) = L$ using ε and δ.

Verify $\lim_{x\to 6^+} x^2 = 6^2$

(2) (optional and harder)

How would you define "$\lim_{x\to\infty} f(x) = L$"

and "$\lim_{x\to c} f(x) = \infty$" mathematically?