期末考解答及评分标准

(1) (10分)

$$\frac{d}{dx}\exp\left(\int_x^{x^2-1}\tan^{-1}(y)\,dy\right)$$

$$= \exp\left(\int_x^{x^2-1}\tan^{-1}(y)\,dy\right)\cdot\frac{d}{dx}\int_x^{x^2-1}\tan^{-1}(y)\,dy \qquad (2分)$$

$$= \exp\left(\int_x^{x^2-1}\tan^{-1}(y)\,dy\right)\cdot\left[\left(\tan^{-1}(x^2-1)\right)\cdot 2x - \tan^{-1}(x)\right] \qquad (8分)$$

(2). (a) (25分)

面积 $= \int_1^4 x\,dx - \int_1^4 (x-2)^2\,dx = \frac{15}{2} - 3 = \frac{9}{2}$ (5分)

周长 = 直线 + 抛物线

$$= 3\sqrt{2} + \int_1^4 \sqrt{1+4(x-2)^2}\,dx \qquad (3分)$$

$$\int_1^4 \sqrt{1+4(x-2)^2}\,dx = \int_{x=1}^{x=4}\sqrt{1+y^2}\,\frac{dy}{2} = \frac{1}{2}\int_{y=-2}^{y=4}\sqrt{1+y^2}\,dy$$

$y = 2(x-2)$

$$\int_{-2}^{4}\sqrt{1+y^2}\,dy = \int_{y=-2}^{y=4}\sec\theta\sec^2\theta\,d\theta = \int_{y=-2}^{y=4}\sec^3\theta\,d\theta \qquad (2分)$$

$y = \tan\theta$

$\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$$\int_{y=-2}^{y=4}\sec^3\theta\,d\theta = \int_{y=-2}^{y=4}\frac{d\theta}{\cos^3\theta} = \int_{y=-2}^{y=4}\frac{\cos\theta\,d\theta}{\cos^4\theta} = \int_{y=-2}^{y=4}\frac{d\sin\theta}{(1-\sin^2\theta)^2}$$

$$= \int_{y=-2}^{y=4}\frac{dz}{(1-z^2)^2} \qquad \rightarrow (5分)$$

$z = \sin\theta$

2(b) (10分)

$$V = \int_1^4 \pi \left(x^2 - [(x-2)^2]^2\right) dx$$

$$= \pi\left(\int_1^4 x^2\,dx - \int_{x=1}^{x=4} (x-2)^4\, d(x-2)\right) = \frac{72}{5}\pi$$

2(c)(25分) 表面积 = 直线部分 + 抛物线部分

直线部分 $= \int_1^4 2\pi x\sqrt{1+1}\,dx = 15\sqrt{2}\pi$ (5分)

抛物线部分:

$$= \int_1^4 2\pi (x-2)^2 \sqrt{1+4(x-2)^2}\,dx$$

$w = x-2$

$$= \int_{w=-1}^{w=2} 2\pi w^2 \sqrt{1+4w^2}\,dw$$

$2w = \tan\theta$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$$= \int_{w=-1}^{w=2} 2\pi \cdot \frac{\tan^2\theta}{2} \sec\theta \cdot \frac{1}{2}\sec^2\theta\,d\theta \quad (5分)$$

$$= \frac{\pi}{2}\int_{w=-1}^{w=2} \frac{\sin^2\theta}{\cos^5\theta}\,d\theta = \frac{\pi}{2}\int_{w=-1}^{w=2} \frac{1-\cos^2\theta}{\cos^5\theta}\,d\theta$$

$$= \frac{\pi}{2}\int_{w=-1}^{w=2}\left(\frac{d\theta}{\cos^5\theta} - \frac{d\theta}{\cos^3\theta}\right) \underset{z=\sin\theta}{=} \frac{\pi}{2}\int_{w=-1}^{w=2} \frac{dz}{(1-z^2)^3} - \frac{dz}{(1-z^2)^2} \quad (5分)$$

$\frac{1}{(1-z^2)^2}$ 部分分式分解：可用未定係數法解

$$\frac{1}{(1-z^2)^2} = \frac{A}{1-z} + \frac{B}{(1-z)^2} + \frac{C}{(1+z)} + \frac{D}{(1+z)^2}$$

或由 $\frac{1}{(1-z)^2} + \frac{1}{(1+z)^2} = \frac{2(1+z^2)}{(1-z)^2(1+z)^2}$

及 $\frac{1}{1-z} + \frac{1}{1+z} = \frac{2}{1-z^2} = \frac{2-(2z^2)}{(1-z^2)^2}$

得 $\frac{1}{(1-z)^2} + \frac{1}{(1+z)^2} + \frac{1}{1-z} + \frac{1}{1+z} = \frac{4}{(1-z^2)^2}$

$$\therefore \frac{1}{(1-z^2)^2} = \frac{1}{4}\left[\frac{1}{1-z} + \frac{1}{1+z} + \frac{1}{(1-z)^2} + \frac{1}{(1+z)^2}\right]$$

$$\therefore \int_{y=2}^{y=4} \frac{1}{(1-z^2)^2}\,dz = \frac{1}{4}\left[-\ln(1-z) + \ln(1+z) - (z-1)^{-1} - (z+1)^{-1}\right]\Bigg|_{y=2}^{y=4}$$

$\begin{cases} y = \tan\theta \\ z = \sin\theta \end{cases}$ ∴ $y=-2 \longleftrightarrow z = \frac{-2}{\sqrt{5}}$ $\left(\begin{array}{l} z=\sin\theta,\ y=\tan\theta \\ y<0 \Leftrightarrow z<0 \end{array}\right.$

$y=4 \longleftrightarrow z = \frac{4}{\sqrt{17}}$

$\Rightarrow y^2+1 = \frac{1}{1-z^2}$

$$\therefore \text{周長} = 3\sqrt{2} + \frac{1}{2}\cdot\frac{1}{4}\left[\ln\left(\frac{1+z}{1-z}\right) - \frac{1}{z-1} - \frac{1}{z+1}\right]\Bigg|_{z=\frac{-2}{\sqrt{5}}}^{z=\frac{4}{\sqrt{17}}} \rightarrow (10\text{分})$$

$$\int \frac{d\theta}{\cos^3\theta} = \int \frac{d\sin\theta}{(1-\sin^2\theta)^3} = \int \frac{dz}{(1-z^2)^3} \qquad z=\sin\theta.$$

可由未定係數法求解

$$\frac{1}{(1-z^2)^3} = \frac{A}{1-z} + \frac{B}{(1-z)^2} + \frac{C}{(1-z)^3} + \frac{D}{1+z} + \frac{E}{(1+z)^2} + \frac{F}{(1+z)^3}$$

或由 $\frac{1}{1-z} + \frac{1}{1+z} = \frac{2}{(1-z)(1+z)} = \frac{2(1-z^2)^2}{(1-z)^3(1+z)^3} = \frac{2(1-2z^2+z^4)}{(1-z)^3(1+z)^3}$

$$\begin{cases} \frac{1}{(1-z)^2} + \frac{1}{(1+z)^2} = \frac{2(1+z^2)}{(1-z)^2(1+z)^2} = \frac{2(1+z^2)(1-z^2)}{(1-z)^3(1+z)^3} = \frac{2(1-z^4)}{(1-z)^3(1+z)^3} \\ \frac{1}{(1-z)^3} + \frac{1}{(1+z)^3} = \frac{2(1+3z^2)}{(1-z)^3(1+z)^3} = \frac{\cancel{\not 0}\,2(\cancel{1})}{(1-z)^3\cancel{(1+z)^3}} \end{cases}$$

得出 $\frac{1}{(1-z^2)^3} = \frac{3}{16}\left[\left(\frac{1}{1-z} + \frac{1}{1+z}\right) + \left(\frac{1}{(1-z)^2} + \frac{1}{(1+z)^2}\right) + \left(\frac{1}{(1-z)^3} + \frac{1}{(1+z)^3}\right)\cdot\frac{2}{3}\right]$

$$\Rightarrow \int_{-\frac{2}{\sqrt{5}}}^{\frac{4}{\sqrt{17}}} \frac{dz}{(1-z^2)^3} = \frac{3}{16}\left[\ln\left(\frac{1+z}{1-z}\right) + \left(\frac{1}{z-1}\right) - \left(\frac{1}{z+1}\right) + \frac{2}{3}\left[\frac{1}{-2}\frac{1}{(z+1)^2} + \frac{1}{2}\frac{1}{(z-1)^2}\right]\right]_{-\frac{2}{\sqrt{5}}}^{\frac{4}{\sqrt{17}}}$$

(10分)

3：每小題(5分)

(a) $\int_1^\infty \frac{1}{\sqrt{1+x^3}}dx$，當 $x\to\infty$ 時 $\frac{1}{\sqrt{1+x^3}} \sim x^{-\frac{3}{2}} \to$ 收斂

$\left(\frac{1}{\sqrt{1+x^3}} < \frac{1}{\sqrt{x^3}}\right)$

(b) $\int_1^\infty \frac{\ln x}{x}dx$ $\quad \frac{\ln x}{x} > \frac{1}{x}$ (當 x 夠大時 $(x>e)$)

而 $\because \int_1^\infty \frac{1}{x}dx$ 發散，故 $\int_1^\infty \frac{\ln x}{x}dx$ 發散.

(c) $\int_{-\infty}^\infty \frac{1}{\cosh x}dx$ ~~當 $|x|\to\infty$ 時~~

$$\cosh x = \frac{e^x+e^{-x}}{2} > \frac{e^{|x|}}{2}$$

故 $\int_{-\infty}^\infty \frac{1}{\cosh x}dx < \int_{-\infty}^\infty \frac{1}{2}e^{-|x|}dx$ 收斂.

4.5：4(a),(b),(c)；5(a),(c),(d) 每小題10分，共60分

4(a) $\int_{-1}^1 \frac{1}{x^2+2x+3}dx = \int_{-1}^1 \frac{dx}{(x+1)^2+2}$

$$= \frac{1}{\sqrt{2}}\int_{-1}^1 \frac{d(x/\sqrt{2})}{1+\left(\frac{x+1}{\sqrt{2}}\right)^2} = \frac{1}{\sqrt{2}}\left(\tan^{-1}\left(\frac{2}{\sqrt{2}}\right) - \tan^{-1}0\right) = \boxed{\frac{1}{\sqrt{2}}\tan^{-1}\sqrt{2}}$$

4(b) $\frac{1}{1-x^4} = \frac{1}{2}\left(\frac{1}{1-x^2}+\frac{1}{1+x^2}\right)$

$$= \frac{1}{2}\left[\left(\frac{1}{1-x}+\frac{1}{1+x}\right)\cdot\frac{1}{2}+\frac{1}{1+x^2}\right]$$

$$= \frac{1}{4}\cdot\frac{1}{1-x}+\frac{1}{4}\cdot\frac{1}{1+x}+\frac{1}{2}\cdot\frac{1}{1+x^2}$$

$\therefore \int \frac{1}{1-x^4}dx = \boxed{\frac{1}{4}\ln\left|\frac{1+x}{1-x}\right| + \frac{1}{2}\tan^{-1}x + C}$

4(c) $\int \frac{\cot x}{1+\sin^2 x}\,dx$

$= \int \frac{\cos x\,dx}{\sin x + \sin^3 x} = \int \frac{dy}{y+y^3}$ $\quad y = \sin x$

$= \int \left(\frac{1}{y} + \frac{-y}{1+y^2}\right) dy = \ln|y| - \frac{1}{2}\ln(1+y^2) + C$

$= \boxed{\ln|\sin x| - \frac{1}{2}\ln(1+\sin^2 x) + C}$

5(a) $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx$ 萬能代換 $t = \tan\frac{x}{2}$, $x = 2\tan^{-1} t$

$\sin x = \frac{2t}{1+t^2}$

$= \int_{x=0}^{x=\frac{\pi}{2}} \frac{1}{1+\frac{2t}{1+t^2}} \frac{2\,dt}{1+t^2}$ $\quad dx = \frac{2\,dt}{1+t^2}$

$= \int_{x=0}^{x=\frac{\pi}{2}} \frac{2}{(t+1)^2}\,dt = -2\,(1+t)^{-1}\Big|_{x=0}^{x=\frac{\pi}{2}}$

$= -2\,\frac{1}{1+t}\Big|_{t=0}^{t=1} = -2\left(\frac{1}{2}-1\right) = \boxed{1}$

5 (c) $\int_0^\infty x^n e^{-x}\,dx = -\int (de^{-x})\,x^n$

$= -e^{-x}x^n\Big|_0^\infty + \int_0^\infty n x^{n-1} e^{-x}\,dx$

$= n\int_0^\infty x^{n-1} e^{-x}\,dx$, by induction, $= n(n-1)\int_0^\infty x^{n-2} e^{-x}\,dx$

$= \cdots = n!\int_0^\infty e^{-x}\,dx = n!$

5(d) $\int \frac{1}{(1+x^2)^2}\,dx$, let $x = \tan\theta$

$= \int \frac{1}{\sec^4\theta} \sec^2\theta\, d\theta$

$= \int \cos^2\theta\, d\theta$

$= \int \frac{1+\cos 2\theta}{2}\, d\theta = \frac{\theta}{2} + \frac{1}{4}\sin 2\theta + C$

$= \frac{\tan^{-1}x}{2} + \frac{1}{4}\sin 2\theta + C$

$= \frac{\tan^{-1}x}{2} + \frac{1}{2}\cdot\frac{x}{(1+x^2)}\cdot\frac{1}{(1+x^2)} + C$